Developing Theories of Types and Computability via Realizability

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Abstract

We investigate the development of theories of types and computability via realizability.

In the first part of the thesis, we suggest a general notion of realizability, based on weakly closed partial cartesian categories, which generalizes the usual notion of realizability over a partial combinatory algebra. We show how to construct categories of so-called assemblies and modest sets over any weakly closed partial cartesian category and that these categories of assemblies and modest sets model dependent predicate logic, that is, first-order logic over dependent type theory. We further characterize when a weakly closed partial cartesian category gives rise to a topos. Scott's category of equilogical spaces arises as a special case of our notion of realizability, namely as modest sets over the category of algebraic lattices. Thus, as a consequence, we conclude that the category of equilogical spaces models dependent predicate logic; we include a concrete description of this model.

In the second part of the thesis, we study a notion of relative computability, which allows one to consider computable operations operating on not necessarily computable data. Given a partial combinatory algebra A, which we think of as continuous realizers, with a subalgebra $A_\sharp \subseteq A$, which we think of as computable realizers, there results a realizability topos $\mathrm{RT}(A, A_\sharp)$, which one intuitively can think of as having "continous objects and computable morphisms". We study the relationship between this topos and the standard realizability toposes $\mathrm{RT}(A)$ and $\mathrm{RT}(A_\sharp)$ over A and $\mathrm{RT}(A_\sharp)$. In particular, we show that there is a localic local map of toposes from $\mathrm{RT}(A, A_\sharp)$ to $\mathrm{RT}(A_\sharp)$. To obtain a better understanding of the relationship between the internal logics of $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A_\sharp)$, we then provide a complete axiomatization of arbitrary local maps of toposes. Based on this axiomatization we investigate the relationship between the internal logics of two toposes connected via a local map. Moreover, we suggest a modal logic for local maps. Returning to the realizability models we show in particular that the modal logic for local maps in the case of $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A_\sharp)$ can be seen as a *modal logic for computability.* Moreover, we characterize some interesting subcategories of $\mathrm{RT}(A, A_\sharp)$ (in much the same way as assemblies and modest sets are characterized in standard realizability toposes) and show the validity of some logical principles in $\mathrm{RT}(A, A_\sharp)$.

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Contents

Chapter 1

Introduction

In this thesis we are concerned with developing theories of types and computability via realizability. Briefly, this means that we develop and study various categories defined via notions of realizability and show how these can be used to model type theories and logics, especially logics to reason about computability. The thesis consists of two parts

Part I A General Notion of Realizability

Part II Local Realizability Toposes and a Modal Logic for Computability

The two parts can be read independently — at the end of this introduction we give a more detailed overview of the dependence of the individual chapters. We now describe the background for our work and overview the results obtained.

1.1 A General Notion of Realizability

Realizability has been used successfully to give models of various logics and type theories in logic and computer science, see, *e.g.*, [Tro98, Hyl82, Lon94, vO99, BCRS98] for many recent references. Typically, the realizers are drawn from some untyped universe, which provides a model of untyped computation for realizers. Examples of such universes of realizers include the natural numbers with Kleene application and models of the untyped lambda calculus (including term models); more generally, any partial combinatory algebra (PCA).

In December 1996, Dana Scott defined the category **Equ** of *equilogical spaces* and showed that it forms a cartesian closed category. In **Equ** objects are topological T_0-spaces with arbitrary equivalence relations on them and morphisms are equivalence classes of equivalence preserving continuous functions. Scott also gave an equivalent presentation of the category **Equ**, namely as partial equivalence relations over the category of algebraic lattices. The category **Equ** is of interest from a computer science perspective since it contains many subcategories of domains (as used in denotational semantics).

In my study of the category **Equ**, I observed that it was advantageous to think of the setup as a generalized form of realizability, where we think of the category of algebraic lattices as providing a *typed* model of computation for realizers. Indeed, in a joint paper with Bauer and Scott [BBS98] we show how **Equ** provides a model of dependent type theory, proceeding by analogy to models constructed over PCA's.

In this thesis we make the analogy precise and suggest a common general framework of which both the model in [BBS98] and also models based on PCA's are instances. Our general notion of realizability is embodied in the definition of a *weakly closed partial cartesian category* (WCPC-category), which is just a weak version of a partial cartesian closed category [RR88]. We prove a couple of results concerning realizability over WCPC-categories; in particular, we characterize when a WCPC-category gives rise to a topos (in a certain way).

We then show how to construct categories of so-called assemblies and modest sets (partial equivalence relations) over any WCPC-category — in the case where the WCPC-category is obtained from a PCA, these categories are the usual categories of assemblies and modest sets over a PCA.

The main results are that the categories of assemblies and modest sets both provide split models of dependent type theory and that they also model a dependent predicate logic with which one can reason about the types and terms in the dependent type theory.

As a consequence, we conclude that **Equ** models dependent predicate logic over dependent type theory. In Appendix A we present this model by writing out explicitly what the interpretation of the calculus is.

1.2 Local Realizability Toposes and a Modal Logic for Computability

In the second part of the thesis we consider a relative notion of computability, developed via realizability. We now describe the motivation for this work, the approach we take, and then give an overview of our results.

1.2.1 Background and Motivation

Suppose we wish to design a new programming language, or extend an existing one, with a richer collection of basic types than is usually found in existing programming languages. For example, we might want to have a type of real numbers (with arbitrary precision) as a basic type, with associated basic operations. Such types contain elements which are not computable since the types contain uncountably many elements. However, it still makes sense to consider *computable operations* on such *not necessarily computable data types*. For instance, the addition of real numbers can be implemented by a computable operation and work correctly also for non-computable real numbers. Of course, we will only want to add to our programming language those operations that are computable, not the rest. To help us decide which types and which operations it makes sense to consider adding to a programming language, we then seek a framework in which we can study computable operations operating on not necessarily computable data types. At the same time, we are naturally interested in a framework with a rich collection of types and with an accompanying logic to reason about the types. We remark that even if one is not interested in programming languages, it is certainly of fundamental interest to have a framework in which to study computability of operations on a wider collection of types than the usual ones.

Scott had the idea that we can consider the usual category $\mathbf{PER}(\mathbb{P})$ of partial equivalence relations over the graph model $\mathbb{P}$ of the lambda calculus and then ask, for a morphism f, whether it is computable by asking whether it is represented by a member of the sub-PCA RE, the recursively enumerable graph model. This idea is a good step in the right direction since the category of partial equivalence relations over $\mathbb{P}$ contains a wide collection of standard mathematical spaces (the category of countably-based T_0-spaces is a full subcategory) and, moreover, asking whether a morphism is computable by asking whether it is represented by a recursively enumerable set is surely a sensible notion of computable. Indeed, one can define a subcategory $\mathbf{PER}(\mathbb{P}, RE)$ of $\mathbf{PER}(\mathbb{P})$ which is full-on-objects and which only has computable morphisms, and this category provides an example of a suitable framework in which to study computable operations on not necessarily computable data. To obtain an expressive logic to reason about the objects and morphisms in $\mathbf{PER}(\mathbb{P}, RE)$, it would be advantageous if this category was a full subcategory of a topos. It is a consequence of my results that this is indeed the case.

Scott also suggested that one can obtain a model of modal logic in the realizability style for reasoning about computability. The idea is to have an extra logical operation $\sharp$ on formulas φ, with formulas φ interpreted by subsets φ of $\mathbb{P}$ and with the formula $\sharp\varphi$ realized by $\varphi \cap RE$, *i.e.*, by computable realizers. Scott showed that $\sharp$, interpreted in this way, satisfies the formal laws for the box operator from S4. It is a consequence of my results that this modal logic may be extended from the propositional case to predicate logic over a wide collection of types such that it can be used to reason about computability of operations operating on not necessarily computable data.

1.2.2 Approach and Overview of Results

Generalizing the ideas described above a bit, we are considering a situation where we have a PCA A, which we think of as the set of continuous realizers, with a sub-PCA $A_\sharp \subseteq A$, which we think of as the computable realizers. There are many other examples besides $\mathbb{P}$ and RE; we describe others later on.

Given A and $A_\sharp$ we consider the standard realizability toposes $\mathrm{RT}(A)$ and $\mathrm{RT}(A_\sharp)$ [HJP80]. Very roughly speaking, we think of $\mathrm{RT}(A)$ as having continuous objects and continuous morphisms, and of $\mathrm{RT}(A_\sharp)$ as having computable objects and computable morphisms. We then identify a third category $\mathrm{RT}(A, A_\sharp)$, which roughly speaking represents the world of continuous objects and computable morphisms. This category $\mathrm{RT}(A, A_\sharp)$ is a topos, the *relative realizability topos* on A with respect to the subalgebra $A_\sharp$.

The toposes $\mathrm{RT}(A)$ and $\mathrm{RT}(A_\sharp)$ are not particularly well-related by themselves; one of the purposes of the relative realizability topos $\mathrm{RT}(A, A_\sharp)$ is to remedy this defect. We show that the three toposes are related to each other as indicated in the following diagram, in which the three functors on the left leg constitute a so-called local geometric morphism, while the right leg is a logical morphism (a filter-quotient).

$$\begin{tikzcd}
& \mathrm{RT}(A, A_\sharp) \arrow[dl] \arrow[dr] & \\
\mathrm{RT}(A_\sharp) \arrow[ur, shift left=2] \arrow[ur, shift right=2] & & \mathrm{RT}(A)
\end{tikzcd}$$

Moreover, we show that the local geometric morphism is in fact localic, that is, $\mathrm{RT}(A, A_\sharp)$ is a localic topos over $\mathrm{RT}(A_\sharp)$.

We thus obtain an understanding of the basic categorical relationship between the three toposes. Since we are chiefly interested in computable morphisms, we then seek to get a deeper understanding of the relationship between the internal logics of the two toposes $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A_\sharp)$. We do so by *axiomatizing* (parts of) the relationship between $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A_\sharp)$. The goal is to put axioms of one of the two toposes such that, given those axioms, one can reconstruct the other topos and the local map between them. The idea is, of course, that such an axiomatization can help in obtaining the better understanding of the relationship between the internal logics we seek.

To make our axiomatization applicable to other examples of local maps (and also to avoid having to make too many detailed and complicated concrete calculations with realizability toposes) we in fact give an elementary axiomatization of *arbitrary* local maps of toposes. Afterwards, we then instantiate our general theory to our particular relative realizability case.

Thus we suggest axioms on a topos $\mathcal{E}$, such that if $\mathcal{E}$ satisfies these axioms, then we can construct a topos $\mathcal{F}$ and show that there is a local geometric morphism from $\mathcal{E}$ to $\mathcal{F}$. In our approach, $\mathcal{E}$ corresponds to $\mathrm{RT}(A, A_\sharp)$ and $\mathcal{F}$ corresponds to $\mathrm{RT}(A_\sharp)$. This approach may be a little bit surprising: it is not based on assuming the existence of an internal locale in a topos, corresponding to $\mathrm{RT}(A_\sharp)$, and then putting axioms on that locale. One advantage of taking the point of view that we take here, that is, the viewpoint corresponding to that of $\mathrm{RT}(A, A_\sharp)$, is that our axiomatization will be more general and not only apply to *localic* local maps but to arbitrary local maps. In a sense, our approach may be seen as analogous to the approach taken in synthetic domain theory (SDT). In SDT a category of domains is singled out abstractly as a full subcategory of a category of general sets. Here we are taking the category $\mathrm{RT}(A, A_\sharp)$ with continuous objects and computable morphisms as given and we are abstractly singling out a full subcategory of "computable objects," namely $\mathrm{RT}(A_\sharp)$. The main result is that our axioms for local maps are sound and complete in the sense that, if the axioms are satisfied, then we indeed get a local map (completeness) and, conversely, given a local map, the axioms are indeed satisfied (soundness).

Based on the axiomatic work, we describe, for any local map $\mathcal{E} \to \mathcal{F}$, the connection between the internal logics of $\mathcal{E}$ and $\mathcal{F}$. Moreover, we derive a modal logic for local maps, which can be used to reason further about the relationship between $\mathcal{E}$ and $\mathcal{F}$.

Since our original example local map is in fact localic, we then specialize our study of the relationships between the internal logics to the case of localic local maps. We show how the modal logic in this case can be phrased in terms of operations on an internal locale. The internal locale is *local* in a obvious sense which we describe in Chapter 9. Moreover, we define a notion of *local tripos* and show that any local tripos gives rise to a localic local map of toposes and that any localic local maps of toposes arise from a local tripos. The approach using local triposes has the advantage over internal locales that it can be easier to recognize a local tripos than a local internal locale, as explained in Chapter 9.

After the abstract study of local maps of toposes and their internal logic, we return to the relative realizability model and show how the modal logic is interpreted there. It turns out that it indeed is a generalization of Scott's original idea of a modal logic for computability mentioned in the previous section. Moreover, we show that the local geometric morphism from $\mathrm{RT}(A, A_\sharp)$ to $\mathrm{RT}(A_\sharp)$ is not open. This result can be seen as partly justifying our choice of axiomatizing local maps and not some smaller class of maps of toposes. We also describe how some of the standard results for realizability toposes concerning the double-negation topology work out in $\mathrm{RT}(A, A_\sharp)$, and we show that $\mathrm{RT}(A, A_\sharp)$ can also be described as the exact completion of a suitable category of partitioned assemblies.

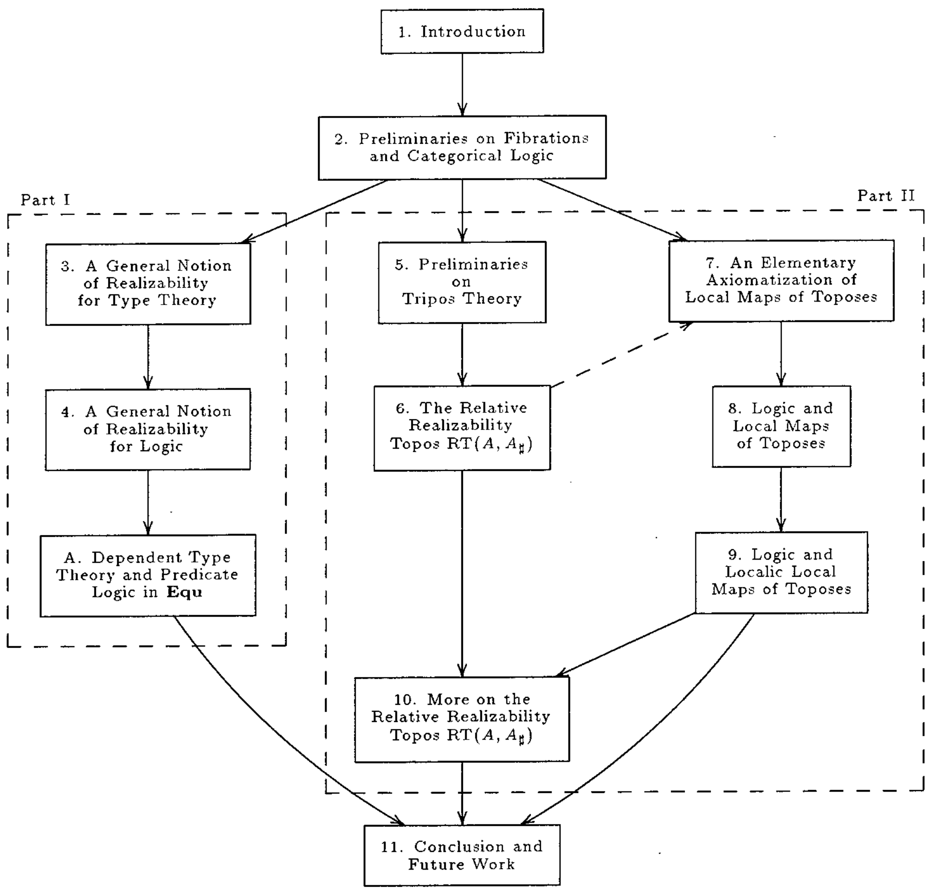
Our work forms part of the research of the Logics of Types and Computation group at Carnegie Mellon University [SAB$^+$]. Indeed we see our work as providing a foundation for more concrete studies of the notion of relative computability. Here we thus stick to the abstract level of a PCA A with a sub-algebra $A_\sharp$ and only occasionally consider more concrete examples. In his forthcoming Ph.D.-thesis [Bau00], Andrej Bauer considers the notion of relative computability in the case of $A = \mathbb{P}$ and $A_\sharp = RE$.

1.2.3 Historical Remarks

I learned about Scott's suggestion of the category of PER's over $\mathbb{P}$ as a suitable category for studying computable operations on not necessarily computable data in 1997. Scott's idea of a modal operator for computability is from January 1998. We learned about the topos $\mathrm{RT}(A, A_\sharp)$ in February 1998 from Thomas Streicher, who suggested it as a suitable framework for studying computable analysis. A couple of months later, however, Martin Hyland was kind enough to let us know that the construction of the topos $\mathrm{RT}(A, A_\sharp)$ has in fact been known for a long time [Pit81, Page 15, item (ii)]. Martin Hyland suggested me to show that $\mathrm{RT}(A, A_\sharp)$ is localic over $\mathrm{RT}(A_\sharp)$ (Hyland new this was the case, see the comments to Theorem 5.4.8) and also to try to axiomatize the local map from $\mathrm{RT}(A, A_\sharp)$ to $\mathrm{RT}(A_\sharp)$. The elementary axiomatization of local maps was carried out jointly with Steven Awodey and some of the results of Part II of the thesis have been published in our joint paper with Awodey and Scott [ABS99].

1.3 Synopsis

The essential dependencies of the various chapters is outlined in the following diagram (the dotted arrow means that Chapter 6 is the motivation for Chapter 7, but mathematically Chapter 7 does not depend on Chapter 6). Part I consists of Chapters 3–4 and Appendix A and Part II consists of Chapters 5–10, as indicated by the dotted boxes. Note that Chapters 7–9 can be read independently of the rest of Part II.



We now outline the content of the remaining chapters.

In **Chapter 2** we recall some of the basic theory of fibrations and categorical logic, which we make use of in the remainder of the thesis.

In **Chapter 3** we begin by recalling some of the theory of categories of partial maps [RR88]. Based on this theory we define the notion of a WCPC-category. We define a notion of pretripos and show that every

WCPC-category gives rise to a pretripos, which can then be used to define categories of so-called assemblies and modest sets over the WCPC-category. We prove that the categories of assemblies and modest sets provide split models of dependent type theory. In the section on realizability pretriposes and universal objects we characterize when a WCPC-category gives rise to a topos and we describe how some of the constructions used in Part I are related to constructions used in Part II of the thesis.

In **Chapter 4** we extend the results of the previous chapter by showing that the categories of assemblies and modest sets over a WCPC-category provide models of dependent predicate logic. We also show how to model subset and quotient types.

In **Chapter 5** we recall some of the theory of triposes, which we shall make use of in the subsequent chapters. We also include a couple of results on triposes, which apparently have not been published before, see Proposition 5.4.7 and Theorem 5.4.8. We recall how the standard realizability tripos over a PCA is defined. Furthermore, we recall the definition of what we term the *relative realizability tripos* over a PCA A with respect to a sub-PCA $A_\sharp$.

In **Chapter 6** we study the relationship between $\mathrm{RT}(A, A_\sharp)$, $\mathrm{RT}(A_\sharp)$, and $\mathrm{RT}(A)$. We prove that there is a localic local geometric morphism from $\mathrm{RT}(A, A_\sharp)$ to $\mathrm{RT}(A_\sharp)$ (Theorem 6.2.3) and show, using results of Pitts, that $\mathrm{RT}(A)$ is a filter-quotient of $\mathrm{RT}(A, A_\sharp)$ (Section 6.1).

In **Chapter 7** we present an elementary axiomatization of local maps of toposes. We recall the definition of a general local map from [Law86, Law89, JM89]. We give an overview of our approach to the axiomatization in Section 7.2 — the approach is based on results of Kelly and Lawvere concerning orthogonal and coorthogonal subcategories and essential localizations of toposes [KL89] — and then proceed to present the axioms after developing a couple of needed definitions and properties. We show that the axioms are sound and complete in a suitable sense (Theorems 7.3.41 and 7.3.44).

In **Chapter 8** we study the relationship between the internal logics of two toposes connected via a local map. In doing so we make use of our axiomatic study from the previous chapter, and we extend, in a way, Lawvere's picture of a local map as an adjoint cylinder (see Chapter 7) to also cover the internal logics. Moreover, we describe a modal logic for local maps. One can think of this modal logic as the internal logic of the given local *map*. We include a couple of examples of applications of the modal logic.

In **Chapter 9** we specialize the treatment of the previous chapter to *localic* local maps. Two additional points of view arise from the assumption that the local map is localic. First we take the point of view of tripos theory and show that the modal logic resulting from the localic local map is just a particular case of tripos logic. We define a notion of *local tripos* and show that any local tripos gives rise to a localic local map of toposes and, moreover, that any localic local map of toposes comes from a local tripos. The actual tripos that results from a localic local map is naturally one given on an internal locale (complete Heyting algebra). Thus we next take the point of view of internal locale theory and describe the modal operators as certain easily given internal maps on an internal locale. We further observe that a substantial part of the modal logic follows from very weak assumptions (whenever one has an internal locale in some topos).

In **Chapter 10** we then finally return to the relative realizability topos $\mathrm{RT}(A, A_\sharp)$. We show how the abstract definitions used in the axiomatic treatment of local maps are instantiated in $\mathrm{RT}(A, A_\sharp)$. Moreover, we show how the modal logic for localic local maps is interpreted via the local map from $\mathrm{RT}(A, A_\sharp)$ to $\mathrm{RT}(A_\sharp)$. We show that the local map from $\mathrm{RT}(A, A_\sharp)$ to $\mathrm{RT}(A_\sharp)$ is not open (so does not preserve all of first-order logic). We also use the chapter to collect some other specific results regarding $\mathrm{RT}(A, A_\sharp)$, including a treatment of the double-negation topology and the fact that $\mathrm{RT}(A, A_\sharp)$ can be seen as an exact completion. Most of these results are simply obtained by verifying that known results for standard realizability toposes can be carried over to the relative realizability setting.

In **Chapter 11** we finally conclude and present some suggestions for future work.

1.4 Prerequisites and Guidelines

We assume familiarity with basic category theory as in [Mac71]. Some acquaintance with dependent type theory, (intuitionistic) logic, and categorical logic will also be useful. For the second part of the thesis we further assume familiarity with basic topos theory [MM92, Joh77] (for the most part, [MM92] suffices) and with (internal) locales / complete Heyting algebras [FS79, Joh82].

Some of the chapter introductions will require more background than the chapter itself; that is, various notions used in the introduction of a chapter will be recalled and defined in the chapter itself. In the beginning of the thesis we will spell out more details than we will towards the end. We should also mention that we do not always refer to the original source of some result; in particular, for background material we seek to refer to easily accessible material. Finally, when we recall standard preliminary material (in particular, in Chapters 2 and 5 and in Section 3.6.1) we state at the beginning of the relevant chapter/section which sources we use; we do not explicitly mark every single recalled definition or result as such.

Chapter 2

Preliminaries on Fibrations and Categorical Logic

In this chapter we recall some background material on fibrations and categorical logic which we use in the sequel. In Section 2.1 we describe our notational conventions for basic category theory and logic. In Section 2.2 we recall the basic definitions and results from fibred category theory that we shall need. Furthermore, we give a very rough sketch of how logics can be interpreted in suitable fibrations and recall a categorical description of logic. Our presentation is based *very closely* on [Jac99] to which the reader is referred to for further background and details. There are several other good introductory sources on fibrations and categorical logic besides [Jac99], see, *e.g.*, [Bén85, Bor94b, Pho93] for material on fibrations and [Her93, Pav90, Tay86, Tay99, Pho93, Cro93] for categorical models of type theory and logic in fibred and indexed categories.

Readers familiar with fibrations and categorical logic as in [Jac99] may skip this chapter.

2.1 Notational Preliminaries

In this section we describe our notational conventions for category theory and logic. We follow [Jac99].

2.1.1 Category Theory

Arbitrary categories are written as $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, ... in open face. Arbitrary toposes are written as $\mathcal{E}$, $\mathcal{F}$, ... in calligraphy. Specific categories, like **Set**, are written in bold face. We generally use capital letters for objects and write $X \in \mathbb{C}$ to express that X is an object of $\mathbb{C}$. We generally use lower case letters for morphisms (also called maps, or arrows) of a category. The homset $\mathbb{C}(X,Y)$ is the collection of morphisms from X to Y in a category $\mathbb{C}$. Unless otherwise stated, categories are assumed to be locally small, that is, with $\mathbb{C}(X,Y)$ a *set* (not a proper class), for all objects X and Y in $\mathbb{C}$. We also sometimes write $\mathrm{Hom}_{\mathbb{C}}(X,Y)$ for $\mathbb{C}(X,Y)$. The notations $f\colon X \to Y$ and $X \xrightarrow{f} Y$ are also used for $f \in \mathbb{C}(X,Y)$. We write $X \rightarrowtail Y$ for monomorphisms and $X \twoheadrightarrow Y$ for epimorphisms. The opposite of a category $\mathbb{C}$ is written $\mathbb{C}^{\mathrm{op}}$ and equivalence of categories $\mathbb{A}$ and $\mathbb{B}$ is written $\mathbb{A} \simeq \mathbb{B}$.

The identity morphism on an object X is written id_X or simply id. Composition of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is written $g \circ f$ or simply gf. A natural transformation between functors $F, G\colon \mathbb{A} \to \mathbb{B}$ is written with a double arrow as $\alpha\colon F \Rightarrow G$. We generally use 1 for a terminal object. Binary products are written $X \times Y$ with projections $\pi\colon X \times Y \to X$ and $\pi'\colon X \times Y \to Y$ and tuples $\langle f, g\rangle\colon Z \to X \times Y$ for $f\colon Z \to X$ and $g\colon Z \to Y$. We often write δ or $\delta(X)$ or δ_X for the diagonal $\langle id, id\rangle\colon X \to X \times X$, and δ or $\delta(I, X)$ for the "parameterized diagonal" $\langle id, \pi'\rangle\colon I \times X \to (I \times X) \times X$, which duplicates X, with parameter I. The exponent object of objects X and Y in a cartesian closed category (CCC) is denoted Y^X or $X \Rightarrow Y$. The evaluation map is written $\mathrm{Ev}\colon Y^X \times X \to Y$ and the abstraction map is written $\Lambda(f)\colon Z \to Y^X$ for $f\colon Z \times X \to Y$.

An initial object is usually denoted 0. For binary coproducts we write $X + Y$ with coprojections $\kappa\colon X \to X + Y$ and $\kappa'\colon Y \to X + Y$ and cotuples $[f, g]\colon X + Y \to Z$ where $f\colon X \to Z$ and $g\colon Y \to Z$.

For functors F and G in an adjunction $F \dashv G$ (F left adjoint to G), the natural isomorphism $\mathbb{B}(FX, Y) \cong \mathbb{A}(X, GY)$ is often written as a bijective correspondence between morphisms $f\colon FX \to Y$ and $g\colon X \to GY$ via double lines:

$$\frac{FX \xrightarrow{f} Y}{\overline{X \xrightarrow[g]{} GY}}$$

The transpose of $f\colon FX \to Y$ is often written as $\hat{f}\colon X \to GY$ and the transpose of $g\colon X \to GY$ is often written as $\check{g}\colon FX \to Y$.

For the rest we follow usual categorical notation, as in the standard reference [Mac71].

2.1.2 Logic

We standardly use many-typed (= many-sorted) logic and we do not restrict ourselves to logic over simple type theory, but also allow logics over dependent type theory. Contexts of variable declarations will be written explicitly at all times, *e.g.*, we write

$$n\colon \mathsf{N} \mid n+5=7 \vdash n=2$$

for a logical entailment. Here $\mid$ is used to separate the type theoretic context $n\colon \mathsf{N}$ from the logical context $n+5=7$. The reason for carrying along these contexts comes from their important categorical rôle as indices (in the fibrational terminology, the context indicates in which fibre we are).

We write $\bot$ for *falsum* (falsehood), $\vee$ for disjunction, $\top$ for truth, $\wedge$ for conjunction, and $\supset$ for implication. Negation $\neg$ will be defined as $\neg\varphi \equiv \varphi \supset \bot$. Existential and universal quantification will be written in typed form $\exists x\colon \sigma.\ \varphi$ and $\forall x\colon \sigma.\ \varphi$. All these proposition formers will be used with their standard rules. Higher-order logic will be described via a distinguished (constant) type $\mathsf{Prop}\colon \mathsf{Type}$, which enables quantification over propositions, as in $\forall\alpha\colon \mathsf{Prop}.\ \varphi$.

Unless otherwise mentioned, logics will always be intuitionistic.

2.2 Preliminaries on Fibrations and Categorical Logic

2.2.1 Fibrations

Let $p\colon \mathbb{E} \to \mathbb{B}$ be a functor. For an object $I \in \mathbb{B}$, the **fibre** or **fibre category** $\mathbb{E}_I$ over I is the category with

objects $X \in \mathbb{E}$ for which $pX = I$.

morphisms $X \to Y$ in $\mathbb{E}_I$ are morphisms $f\colon X \to Y$ in $\mathbb{E}$ for which $pf = id_I$.

An object $X \in \mathbb{E}$ satisfying $pX = I$ is said to be **above** I; similarly, a morphism f in $\mathbb{E}$ with $pf = u$ is said to be **above** u. A morphism in $\mathbb{E}$ is said to be **vertical** if it is above some identity morphism in $\mathbb{B}$. For $X, Y \in \mathbb{E}$ and $u\colon pX \to pY$ in $\mathbb{B}$, we sometimes write

$$\mathbb{E}_u(X,Y) = \{\, f\colon X \to Y \text{ in } \mathbb{E} \mid f \text{ is above } u \,\} \subseteq \mathbb{E}(X,Y).$$

Definition 2.2.1. Let $p\colon \mathbb{E} \to \mathbb{B}$ be a functor.

1. A morphism $f\colon X \to Y$ is **cartesian over** $u\colon I \to J$ if $pf = u$ and if every $g\colon Z \to Y$ in $\mathbb{E}$ for which one has $pg = u \circ w$ for some $w\colon pZ \to I$, uniquely determines an $h\colon Z \to X$ in $\mathbb{E}$ above w with $f \circ h = g$. In a diagram:

$$\begin{tikzcd} Z \arrow[dr, dashed, "h"'] \arrow[drr, "g"] & & \\ & X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \text{in } \mathbb{E}$$

$$\begin{tikzcd} pZ \arrow[dr, "w"'] \arrow[drr, "pg"] & & \\ & I \arrow[r, "u"'] & J \end{tikzcd} \qquad \text{in } \mathbb{B}.$$

   Since a morphism $f\colon X \to Y$ can only be cartesian over its underlying map pf in $\mathbb{B}$, we just call f **cartesian** if this is the case.

2. The functor $p\colon \mathbb{E} \to \mathbb{B}$ is a **fibration** if for every $Y \in \mathbb{E}$ and $u\colon I \to pY$ in $\mathbb{B}$, there is a cartesian morphism $f\colon X \to Y$ in $\mathbb{E}$ above u. Sometimes a fibration will be called a **fibred category** or a **category (fibred) over** $\mathbb{B}$.

We often write $\begin{array}{c}\mathbb{E}\\ \downarrow{\scriptstyle p}\\ \mathbb{B}\end{array}$ for a fibration $p\colon \mathbb{E} \to \mathbb{B}$ and refer to $\mathbb{E}$ as the **total category** and to $\mathbb{B}$ as the **base category**. When the functor p is clear from context, we often simply write $\begin{array}{c}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{array}$ (pronounced "$\mathbb{E}$ over $\mathbb{B}$").

We often say that a cartesian morphism $f\colon X \to Y$ over $u\colon I \to pY$ is a **cartesian lifting** of u. Cartesian liftings are unique up-to-isomorphism: if f and f' are both cartesian over the same map, then there is a unique vertical isomorphism φ with $f' \circ \varphi = f$ (indeed we have that $\mathbb{B}/J(p(-), u) \cong \mathbb{E}/Y(-, f)$).

We write $\mathbb{B}^{\to}$ for the **arrow category** of $\mathbb{B}$ with

objects morphisms $\varphi\colon X \to I$ in $\mathbb{B}$.

morphisms $(\varphi\colon X \to I) \to (\psi\colon Y \to J)$ are pairs of morphisms (u, f) with $u\colon I \to J$ and $f\colon X \to Y$ for which the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

commutes.

We write $\mathrm{cod}\colon \mathbb{B}^{\to} \to \mathbb{B}$ for the codomain functor. As the reader can verify, the functor cod is a fibration if and only if $\mathbb{B}$ has pullbacks. We refer to it as the **codomain fibration** on $\mathbb{B}$. The fibre category over I is the slice category $\mathbb{B}/I$ and cartesian morphisms in $\mathbb{B}^{\to}$ coincide with pullback squares in $\mathbb{B}$.

We write Mono ($\mathbb{B}$) for the full subcategory of $\mathbb{B}^{\to}$ on the objects $\varphi\colon X \to I$ which are monomorphisms in $\mathbb{B}$. If $\mathbb{B}$ has pullbacks, then the restricted codomain functor $\begin{smallmatrix}\mathrm{Mono}(\mathbb{B})\\ \downarrow \\ \mathbb{B}\end{smallmatrix}$ is again a fibration since the pullback of a monomorphism is a monomorphism. Note that all the fibres of Mono ($\mathbb{B}$) are preordered categories. Such a fibration for which all the fibres are preorders, will be called a **fibred preorder**.

We write Sub($\mathbb{B}$) for the category obtained from Mono ($\mathbb{B}$) by taking subobjects as objects. If $\mathbb{B}$ has pullbacks, $\begin{smallmatrix}\mathrm{Sub}(\mathbb{B})\\ \downarrow \\ \mathbb{B}\end{smallmatrix}$ is referred to as the **fibration of subobjects** or the **subobject fibration** of $\mathbb{B}$. The fibres Sub(I) over objects $I \in \mathbb{B}$ are partial orders.

Cloven and Split Fibrations

Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p \\ \mathbb{B}\end{smallmatrix}$ be a fibration. Then for each $u\colon I \to J$ in the base $\mathbb{B}$ and each $X \in \mathbb{E}$ above J, there is a cartesian lifting $\bullet \to X$. Assume now that we *choose* for each such u a specific cartesian lifting and write it as

$$\overline{u}(X)\colon u^*(X) \to X.$$

Having made such choices, every map $u\colon I \to J$ in $\mathbb{B}$ determines a functor $u^*\colon \mathbb{E}_J \to \mathbb{E}_I$. For an object $X \in \mathbb{E}_J$, one takes $u^*(X)$ to be the domain of the chosen cartesian lifting $\overline{u}(X)\colon u^*(X) \to X$. For a map $f\colon X \to Y$ in $\mathbb{E}_J$, one takes $u^*(f)$ to be the unique map from $u^*(X)$ to $u^*(Y)$ with

$\overline{u}(Y) \circ u^*(f) = f \circ \overline{u}(X)$. Such functors u^* are referred to as **reindexing functors**, **substitution functors** , **relabelling functors** , or sometimes also as **change-of-base functors** or **pullback functors** . We mostly use the first two names.

For two composable morphisms

$$I \xrightarrow{u} J \xrightarrow{v} K$$

in $\mathbb{B}$, in general one does not have equality $u^*v^* = (v \circ u)^*$ but only a natural isomorphism

$$u^*v^* \Rightarrow (v \circ u)^*. \tag{2.1}$$

Likewise, there are natural isomorphisms

$$id \Rightarrow (id)^*. \tag{2.2}$$

These natural transformations satisfy certain coherence conditions, but we shall not go into that here [Jac99].

Definition 2.2.2.

1. A fibration is called **cloven** if it comes equipped with a **cleavage**, that is, with a choice of cartesian liftings. This cleavage then induces substitution functors u^* between the fibres, as above.

2. A fibration is called **split** if the induced substitution functors are such that the canonical natural transformations in (2.1) and (2.2) are identities:

$$id = (id)^* \qquad \text{and} \qquad u^*v^* = (v \circ u)^*.$$

   The cleavage involved is then often called a **splitting**.

If $\mathbb{B}$ is a category with *chosen* pullbacks, then the codomain fibration on $\mathbb{B}$ is cloven, but in general it is not split. The subobject fibration, on the other hand, is trivially split since the fibres are partial orders. We shall see more examples of split fibrations in Chapter 3.

Indexed Categories

Definition 2.2.3.

1. A **$\mathbb{B}$-indexed category** is a **pseudo functor** $\Psi\colon \mathbb{B}^{op} \to \mathbf{Cat}$. It consists of a mapping which assigns to each object $I \in \mathbb{B}$ a category $\Psi(I)$ and to each morphism $u\colon I \to J$ a functor $\Psi(u)\colon \Psi(J) \to \Psi(I)$, often simply denoted u^* when no confusion arises. Additionally, a pseudo-functor involves natural isomorphisms

$$\begin{array}{ll} \eta_I\colon id \Rightarrow (id_I)^* & \text{for } I \in \mathbb{B} \\ \mu_{u,v}\colon u^*v^* \Rightarrow (v \circ u)^* & \text{for } I \xrightarrow{u} J \xrightarrow{v} K \text{ in } \mathbb{B} \end{array}$$

which satisfy the coherence conditions:

$$\begin{tikzcd} & u^* \arrow[dl, "\eta_I u^*"'] \arrow[d, equal] \arrow[dr, "u^*\eta_J"] & \\ (id_I)^*u^* \arrow[r, "\mu_{id_I,u}"'] & u^* & u^*(id_J)^* \arrow[l, "\mu_{u,id_J}"] \end{tikzcd} \qquad \text{for } I \xrightarrow{u} J$$

$$\begin{tikzcd} u^*v^*w^* \arrow[r, "u^*\mu_{v,w}"] \arrow[d, "\mu_{u,v}w^*"'] & u^*(w \circ v)^* \arrow[d, "\mu_{u,w\circ v}"] \\ (v \circ u)^*w^* \arrow[r, "\mu_{v\circ u,w}"'] & (w \circ v \circ u)^* \end{tikzcd} \qquad \text{for } I \xrightarrow{u} J \xrightarrow{v} K \xrightarrow{w} L$$

2. A **split (also called strict) $\mathbb{B}$-indexed category** is just a functor $\Psi\colon \mathbb{B}^{op} \to \mathbf{Cat}$; it is an indexed category for which the η's and μ's in item 1 are identities.

Proposition 2.2.4. *Let* $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ *be a fibration with a cleavage. The assignment*

$$I \mapsto \mathbb{E}_I \qquad \textit{and} \qquad u \mapsto \textit{(the substitution functor } u^*\textit{)}$$

determines a $\mathbb{B}$-indexed category. This indexed category is split whenever the cleavage of p is a splitting.

Definition 2.2.5 (Grothendieck construction). Let $\Psi\colon \mathbb{B}^{op} \to \mathbf{Cat}$ be an indexed category. The **Grothendieck completion** $\int_{\mathbb{B}} \Psi$ (or simply $\int \Psi$) of Ψ is the category with

objects (I, X) where $I \in \mathbb{B}$ and $X \in \Psi(I)$.

morphisms $(I, X) \to (J, Y)$ are pairs (u, f) with $u\colon I \to J$ in $\mathbb{B}$ and $f\colon X \to u^*(Y) = \Psi(u)(Y)$ in $\Psi(I)$.

Composition and identities in $\int \Psi$ involve the isomorphisms η and μ from Definition 2.2.3. The identity on (I, X) is the pair $(id, \eta_I(X))$, where

$$\eta_I \colon id_{\Psi(I)} \Rightarrow (id_I)^*.$$

And composition in $\int \Psi$ of

$$(I, X) \xrightarrow{(u,f)} (J, Y) \xrightarrow{(v,g)} (K, Z)$$

is defined as

$$\begin{cases} I \xrightarrow{u} J \xrightarrow{v} K \\ X \xrightarrow[f]{} u^*(Y) \xrightarrow[u^*(g)]{} u^*v^*(Z) \xrightarrow[\mu_{u,v}(Z)]{\cong} (v \circ u)^*(Z) \end{cases}$$

The required equalities for identity and composition follow from the coherence diagrams in Definition 2.2.3. In fact, these conditions capture precisely what is required for $\int \Psi$ to be a category.

Proposition 2.2.6. *Let $\Psi \colon \mathbb{B}^{\mathrm{op}} \to \mathbf{Cat}$ be a $\mathbb{B}$-indexed category.*

1. *The first projection $\begin{array}{c}\int \Psi \\ \downarrow \\ \mathbb{B}\end{array}$ is a cloven fibration. It is split whenever Ψ is split.*
2. *Turning a cloven fibration into an indexed category (as in Proposition 2.2.4) and then again into a fibration yields a fibration which is equivalent to the original one.*
3. *Moreover, turning an indexed category first into a fibration and then into an indexed category yields a result which is "essentially the same" as the original (in a sense which can be made precise).*

Change-of-base for Fibrations

Lemma 2.2.7. *Let $\begin{array}{c}\mathbb{E} \\ \downarrow p \\ \mathbb{B}\end{array}$ be a fibration and let $K \colon \mathbb{A} \to \mathbb{B}$ be a functor. Form the pullback in* **Cat**

$$\begin{tikzcd} \mathbb{A} \times_{\mathbb{B}} \mathbb{E} \arrow[r] \arrow[d, "K^*(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{A} \arrow[r, "K"'] & \mathbb{B}. \end{tikzcd}$$

In this situation, the functor $K^*(p)$ is also a fibration. It is cloven or split in case p is cloven or split.

Here we are using ordinary pullbacks of categories: $\mathbb{A} \times_{\mathbb{B}} \mathbb{E}$ has pairs $(I \in \mathbb{A}, X \in \mathbb{E})$ with $KI = pX$ as objects.

Proof. Given an object $(J, Y) \in \mathbb{A} \times_{\mathbb{B}} \mathbb{E}$ and a morphism $u \colon I \to J$ in $\mathbb{A}$, let $f \colon X \to Y$ be the cartesian lifting of $Ku \colon KI \to KJ$ in $\mathbb{B}$. The pair (u, f) is then $K^*(p)$-cartesian over u. $\square$

Categories of Fibrations

Let $\begin{array}{c}\mathbb{E}\\ \downarrow{\scriptstyle p}\\ \mathbb{B}\end{array}$ and $\begin{array}{c}\mathbb{D}\\ \downarrow{\scriptstyle q}\\ \mathbb{B}\end{array}$ be two fibrations over $\mathbb{B}$. A **fibred functor** from p to q is a functor $H : \mathbb{E} \to \mathbb{D}$ such that the diagram

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "H"] \arrow[dr, "p"'] & & \mathbb{D} \arrow[dl, "q"] \\
& \mathbb{B} &
\end{tikzcd}$$

commutes and, moreover, such that H preserves cartesian morphisms.

Fibrations over $\mathbb{B}$ and fibred functors among such constitute a 2-category $\mathbf{Fib}(\mathbb{B})$ with 2-cells natural transformations τ between fibred functors for which every component of τ is vertical.

A fibred functor among split fibrations over $\mathbb{B}$ is **split** if it is preserves the splitting on-the-nose.

If $H : \mathbb{E} \to \mathbb{D}$ is a fibred or split fibred functor as above, then for each object $I \in \mathbb{B}$, one obtains by restriction a functor $H_I \colon \mathbb{E}_I \to \mathbb{D}_I$ between the fibres over I. For $u \colon I \to J$ in $\mathbb{B}$, writing u^* for the reindexing functor $\mathbb{E}_J \to \mathbb{E}_I$ and $u^\sharp$ for the reindexing functor $\mathbb{D}_J \to \mathbb{D}_I$, one has that $u^\sharp H_J \cong H_I u^*$.

Fibrewise Structure and Fibred Adjunctions

Definition 2.2.8. Let $\diamond$ be some categorical property or structure (*e.g.*, some limit, colimit, or exponent).

1. We say a fibration has **fibred** $\diamond$'s or **fibrewise** $\diamond$'s if all fibre categories have $\diamond$'s and reindexing functors preserve $\diamond$'s. A split fibration has **split fibred** $\diamond$'s if all fibres have chosen $\diamond$'s and the reindexing functors induced by the splitting preserve $\diamond$'s on-the-nose.

2. A fibred functor H from $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ to $\begin{array}{c}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{array}$ preserves $\diamond$'s if for each $I \in \mathbb{B}$ the functor H_I preserves $\diamond$'s. For the split version, one requires preservation on-the-nose.

For example, for a category $\mathbb{B}$ with finite limits, the codomain fibration $\begin{array}{c}\mathbb{B}^{\rightarrow}\\ \downarrow\\ \mathbb{B}\end{array}$ always has fibred finite limits. The subobject fibration on such $\mathbb{B}$ has *split* fibred finite limits. A category $\mathbb{B}$ is locally cartesian closed (*i.e.*, all slices are cartesian closed) if and only if the codomain fibration on $\mathbb{B}$ is fibred cartesian closed (*i.e.*, has fibred finite products and exponents).

The following definition and the following three lemmas express that a fibred categorical notion is a property of all fibres, preserved by reindexing.

Definition 2.2.9.

1. An adjunction between fibrations over the same base $\mathbb{B}$ is an adjunction in the 2-category $\mathbf{Fib}(\mathbb{B})$. Explicitly, let $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ and $\begin{array}{c}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{array}$ be fibrations over $\mathbb{B}$. Then a **fibred adjunction** over $\mathbb{B}$ is given by fibred functors F, G as in

$$\begin{tikzcd} \mathbb{E} \arrow[rr, bend left, "F"] \arrow[dr, "p"'] & & \mathbb{D} \arrow[ll, bend left, "G"] \arrow[dl, "q"] \\ & \mathbb{B} & \end{tikzcd}$$

together with *vertical* natural transformations

$$\eta\colon id_{\mathbb{E}} \Rightarrow GF \qquad \text{and} \qquad \epsilon\colon FG \Rightarrow id_{\mathbb{D}}$$

satisfying the usual triangular identities $G\epsilon \circ \eta G = id$ and $\epsilon F \circ F\eta = id$.

2. A **split fibred adjunction** over $\mathbb{B}$ between split fibrations p and q consists of a fibred adjunction as above for which the functors F and G are split.

Lemma 2.2.10. *A fibration* $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ *has a fibred terminal object if and only if the unique morphism from* p *to the terminal object in* $\mathbf{Fib}(\mathbb{B})$ *has a fibred* *right adjoint, say* 1, *as in*

$$\begin{tikzcd} \mathbb{E} \arrow[rr, bend left, "!=p"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathbb{B} \arrow[ll, bend left, "1"] \arrow[dl, "id_{\mathbb{B}}"] \\ & \mathbb{B}. & \end{tikzcd}$$

Lemma 2.2.11. *Let* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *and* $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{smallmatrix}$ *be fibrations over* $\mathbb{B}$ *and let* $H\colon \mathbb{E} \to \mathbb{D}$ *be a fibred functor from* p *to* q. *Then* H *has a fibred left (resp. right) adjoint iff both*

1. *For each object* $I \in \mathbb{B}$, *the functor* $H_I\colon \mathbb{E}_I \to \mathbb{D}_I$ *has a left (resp. right) adjoint* $K(I)$.
2. *The* ***Beck-Chevalley condition*** *holds,* i.e., *for every map* $u\colon I \to J$ *in* $\mathbb{B}$ *and for every pair of reindexing functors*

$$\mathbb{E}_J \xrightarrow{u^*} \mathbb{E}_I \qquad\qquad \mathbb{D}_J \xrightarrow{u^\#} \mathbb{D}_I$$

the canonical natural transformation[1]

$$K(I)u^\# \Rightarrow u^*K(J) \qquad\qquad (\textit{resp. } u^*K(J) \Rightarrow K(I)u^\#)$$

is an isomorphism.

Lemma 2.2.12. *Let* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *and* $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{smallmatrix}$ *be split fibrations over* $\mathbb{B}$ *and let* $H\colon \mathbb{E} \to \mathbb{D}$ *be a split fibred functor. Then* H *has a split fibred left (resp. right) adjoint iff both items 1 and 2 in the previous lemma hold, but in item 2 with the canonical map being an* identity.

Two fibrations $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{smallmatrix}$ over the same base $\mathbb{B}$ are **equivalent** if there are fibred functors $F\colon \mathbb{E} \to \mathbb{D}$ and $G\colon \mathbb{D} \to \mathbb{E}$ over $\mathbb{B}$ with vertical natural isomorphisms $GF \cong id_{\mathbb{E}}$ and $FG \cong id_{\mathbb{D}}$.

---

[1] Obtained as the transpose of $u^\# X \xrightarrow{u^\#\eta} u^\# H_J K_J X \cong H_I(u^* K_J X)$ across the adjunction $K(I) \dashv H_I$. In the sequel we do not write out explicitly how such canonical natural transformations are obtained.

Fibred Products and Coproducts

Definition 2.2.13. Let $\mathbb{B}$ be a category with cartesian products and let $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ be a fibration. We say that p has **simple products** (resp. **simple coproducts**) if both

- for every pair of objects $I, J \in \mathbb{B}$, every "weakening functor"

$$\mathbb{E}_I \xrightarrow{\pi^*_{I,J}} \mathbb{E}_{I\times J}$$

  induced by the projection $\pi_{I,J}\colon I \times J \to I$, has a right adjoint $\prod_{(I,J)}$ (resp. a left adjoint $\coprod_{(I,J)}$).

- the Beck-Chevalley condition holds: for every $u\colon K \to I$ in $\mathbb{B}$ and $J \in \mathbb{B}$ in the diagram

$$\begin{tikzcd}
\mathbb{E}_I \arrow[r, "u^*"] \arrow[d, bend right, "\pi^*_{I,J}"'] & \mathbb{E}_K \arrow[d, bend right, "\pi^*_{K,J}"'] \\
\mathbb{E}_{I\times J} \arrow[r, "(u\times id)^*"'] \arrow[u, bend right] & \mathbb{E}_{K\times J} \arrow[u, bend right]
\end{tikzcd}$$

  the canonical natural transformation

$$u^* \textstyle\prod_{(I,J)} \Rightarrow \prod_{(K,J)}(u \times id)^*$$
$$(\text{resp. } \textstyle\coprod_{(K,J)}(u \times id)^* \Rightarrow u^* \coprod_{(I,J)})$$

  is an isomorphism.

Definition 2.2.14. Let $\mathbb{B}$ be a category with pullbacks and let $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ be a fibration. We say that p has **products** (resp. **coproducts**) if both

- for every morphism $u\colon I \to J$, every substitution functor u^* has a right adjoint $\prod_u$ (resp. left adjoint $\coprod_u$)

- the Beck-Chevalley condition holds: for every pullback in $\mathbb{B}$ of the form

$$\begin{tikzcd}
K \arrow[r, "v"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "s"] \\
I \arrow[r, "u"'] & J
\end{tikzcd}$$

the canonical natural transformation

$$s^* \prod_u \Rightarrow \prod_v r^* \qquad\qquad (\text{resp. } \coprod_v r^* \Rightarrow s^* \coprod_u)$$

is an isomorphism.

Clearly, if a fibration has products (resp. coproducts), then it also has simple products (resp. coproducts).

A split fibration has **split (simple) products** (resp. **split (simple) coproducts** if it has (simple) products and the isomorphism mentioned in the Beck-Chevalley condition is an identity (for the adjoints to the reindexing functors induced by the splitting).

Remark 2.2.15. The Beck-Chevalley condition in the definition of (simple) products comes from the fact that (simple) products are an instance of fibred adjunctions [Her93], which, by Lemma 2.2.11, are described equivalently via adjunctions among the fibres and a Beck-Chevalley condition.

Lemma 2.2.16. *Consider a fibration for which each reindexing functor has both a left* $\coprod$ *and a right adjoint* $\prod$*. Then Beck-Chevalley holds for coproducts* $\coprod$ *iff it holds for products* $\prod$*.*

For a category $\mathbb{B}$ with finite limits, the codomain fibration $\begin{smallmatrix}\mathbb{B}^{\to}\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$ on $\mathbb{B}$ has

1. coproducts $\coprod_u$ given by composition: $\coprod_u(\varphi\colon Y \to J) = \varphi \circ u$
2. simple products $\prod_{(I,J)}$ iff $\mathbb{B}$ is cartesian closed
3. products $\prod_u$ iff $\mathbb{B}$ is locally cartesian closed.

Definition 2.2.17. A fibration is called **complete** if it has products $\prod_u$ and fibred finite limits. Dually, a fibration is **cocomplete** if it has coproducts $\coprod_u$ and fibred finite colimits.

The following lemma will be used in the categorical description of logics and type theories. In logic it corresponds to the equivalence of $\exists x\colon \sigma.\ (\varphi \wedge \psi(x))$ and $\varphi \wedge \exists x\colon \sigma.\ \psi(x)$, if x does not occur free in φ.

Lemma 2.2.18. *Let* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *be a fibred cartesian closed category.*

1. *Suppose p has simple coproducts. For each pair of objects $I, J \in \mathbb{B}$ and each pair of objects $Y \in \mathbb{E}_I$, $Z \in \mathbb{E}_{I\times J}$, the canonical morphism (the Frobenius map)*

$$\coprod_{(I,J)}(\pi_{I,J}^*(Y) \times Z) \longrightarrow Y \times \coprod_{(I,J)}(Z)$$

is an isomorphism.

2. *Suppose now p has coproducts. Then for each $u \colon I \to J$ in $\mathbb{B}$, $Y \in \mathbb{E}_J$ and $Z \in \mathbb{E}_I$, the canonical morphism (the Frobenius map)*

$$\coprod_u(u^*(Y) \times Z) \longrightarrow Y \times \coprod_u(Z)$$

is an isomorphism.

Even if there are no fibred exponents around, the Frobenius map can still be an isomorphism. In that case we shall speak of **(simple) coproducts with the Frobenius property**, or of **(simple) coproducts satisfying Frobenius**.

2.2.2 Categorical Logic

In the sequel we shall make use of fibrations and indexed categories to describe models of logics and type theories. Moreover, we shall often assume given a fibration with certain properties, and then use its internal logic to make new constructions and prove properties.

An interpretation of a logical theory in a given fibration is formally defined by a kind of *functorial semantics*, namely as a morphism of fibrations from a certain *classifying fibration* of the logic to the given fibration. See [Jac99] for a precise detailed treatment. Here we just sketch the general idea and include a description of fibred equality.

The general idea for interpreting many-sorted logic in a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is as follows. Types and terms are interpreted as objects and morphisms in the base category. Contexts Γ are interpreted as objects I of the base category (*e.g.*, as the product of all the types of the variables in the context). A formula $\Gamma \vdash \varphi\colon \mathsf{Prop}$ in context Γ is interpreted as an object in $\mathbb{E}_I$, the fibre over the object I interpreting the context Γ. Substitution of a term in a formula is interpreted by reindexing the interpretation of the formula along the map in the base interpreting the term. Thus, if we have a formula

$$x\colon \sigma \vdash \varphi\colon \mathsf{Prop}$$

and a term

$$\Gamma \vdash M : \sigma$$

with

- Γ interpreted by I
- σ interpreted by J
- $x : \sigma \vdash \varphi : \mathsf{Prop}$ interpreted by $X \in \mathbb{E}_J$
- $\Gamma \vdash M : \sigma$ interpreted by a morphism $m : I \to J$

then the formula φ with M substituted for x, that is,

$$\Gamma \vdash \varphi[M/x] : \mathsf{Prop}$$

is interpreted by $m^*(X)$ in the fibre over I.

When $\Gamma, x : \sigma \vdash \varphi : \mathsf{Prop}$ is a formula, and $\Gamma \vdash M : \sigma$ is a term, we sometimes simply write $\Gamma \vdash \varphi(M) : \mathsf{Prop}$ for $\Gamma \vdash \varphi[M/x] : \mathsf{Prop}$.

Entailment $\Gamma \mid \varphi \vdash \psi$ is interpreted as the existence of an arrow from the interpretation of φ to the interpretation of ψ in $\mathbb{E}_I$ (where I is again the interpretation of Γ). Since in logic one does not typically distinguish between different proofs of the same entailment, fibrations for interpreting logics will typically be preorder fibrations. More general fibrations will be used to model type theories (where one does distinguish between different terms).

Fibred Equality

We recall the categorical description of equality in terms of left adjoints to **contraction functors** δ^*. The approach is due to Lawvere [Law68]; we follow the presentation in [Jac99].

In a category with products we write, for objects I and J,

$$I \times J \xrightarrow{\delta = \delta(I,J) = \langle id, \pi' \rangle} (I \times J) \times J$$

for the "parameterized" diagonal which duplicates J, with parameter I.

Definition 2.2.19. Let $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ be a fibration on a base category $\mathbb{B}$ with finite products.

1. Then p is said to have **(simple) equality** if both
   - for every pair $I, J \in \mathbb{B}$, each contraction functor $\delta(I, J)^*$ has a left adjoint
   $$\mathbb{E}_{I \times J} \xrightarrow{\mathrm{Eq}_{I,J} = \coprod_{\delta(I,J)}} \mathbb{E}_{(I \times J) \times J}$$
   - the Beck-Chevalley condition holds: for each map $u \colon K \to I$ in $\mathbb{B}$, the canonical natural transformation
   $$\mathrm{Eq}_{K,J}(u \times id)^* \Rightarrow ((u \times id) \times id)^* \mathrm{Eq}_{I,J}$$
   is an isomorphism.
2. If p is a fibration with fibred finite products $\times$, then we say that p has **equality with the Frobenius property** or **equality satisfying Frobenius** if it has equality as described above in such a way that for all objects $X \in \mathbb{E}_{(I \times J) \times J}$ and $Y \in \mathbb{E}_{I \times J}$, the canonical map
$$\mathrm{Eq}_{I,J}(\delta^*(X) \times Y) \longrightarrow X \times \mathrm{Eq}_{I,J}(Y)$$
is an isomorphism.

Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration with equality. Assume that p has a terminal object functor $1 \colon \mathbb{B} \to \mathbb{E}$, which for each $I \in \mathbb{B}$ gives the terminal object in the fibre over I. For parallel maps $u, v \colon I \to J$ in $\mathbb{B}$ we write
$$\mathrm{Eq}(u, v) \stackrel{\mathrm{def}}{=} \langle\langle id, u\rangle, v\rangle^*(\mathrm{Eq}_{I,J}(1)) \in \mathbb{E}_I,$$
where $1 = 1(I \times J)$ is the terminal object in the fibre over $I \times J$. This yields an equality predicate in the fibre over the domain I of the maps u and v. We think intuitively of the predicate $\mathrm{Eq}(u, v)$ at $i \in I$ as the truth of $u(i) =_J v(i)$ in the "internal logic of the fibration p". Formally, we say that $u, v \colon I \to J$ are **internally equal** if there is a morphism (a "proof") $1 \to \mathrm{Eq}(u, v)$ in the fibre over I. This need not be the same as **external equality** of u, v which simply means that $u = v$ as morphisms of $\mathbb{B}$. External equality always implies internal equality. In case internal equality is the same as external equality, we say that the fibration has **very strong equality**.

Let $\mathbb{B}$ be a category with finite limits and consider the subobject fibration $\begin{smallmatrix}\mathrm{Sub}(\mathbb{B})\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$ on $\mathbb{B}$. For maps $u, v \colon I \to J$ in $\mathbb{B}$, one can easily verify that $\mathrm{Eq}(u, v)$ is the equalizer of u and v. Subobject fibrations therefore always have very strong equality. See [Jac99, Section 3.5] for some examples of fibrations which do not have very strong equality.

Part I

A General Notion of Realizability

Chapter 3

A General Notion of Realizability for Type Theory

We define a general notion of realizability to be a *weakly closed partial cartesian category* (WCPC-category), which encompasses both partial combinatory algebras and algebraic lattices (and many more), see Section 3.1. Using this notion of realizability we show how to construct categories of so-called assemblies and modest sets (partial equivalence relations) over them. We then also show that the so constructed categories provide models of dependent type theory. Throughout the chapter we focus mostly on assemblies; then in Section 3.7 we briefly show how to obtain corresponding results for modest sets. The main result of the chapter is Theorem 3.6.20, which says that the category of assemblies provides a split model of dependent type theory. We also characterize when a WCPC-category gives rise to a topos. We now provide an outline of the chapter.

Rather than going straight from the general notion of realizability to the category of assemblies there-over, we proceed in stages, making use of ideas and notions from categorical logic. This approach has the advantage that it allows us to state and prove results at a more general level. Moreover, we emphasize the connection to tripos theory [HJP80, Pit81], which we shall make use of in later chapters. Furthermore, we shall see that in the construction of models for dependent type theory we can reason abstractly using the internal logic of a realizability pretripos; we do not need to go into detailed manipulation of realizers.

Thus in Section 3.3, after recalling in Section 3.2 the notions of regular categories and regular fibrations, we show how to define a category of assemblies over any regular fibration and we show that the so-defined category of assemblies is regular.

In Section 3.4 we define a notion of pretripos, which is a weak version of tripos [HJP80, Pit81]. We further show that any WCPC-category gives rise to a pretripos over **Set**. We also characterize precisely when a WCPC-category gives rise to a topos; this is the case iff the WCPC-category has a so-called universal object of which all other objects are retracts.

A pretripos is a regular fibration and in Section 3.5 we show that the category of assemblies over a pretripos is locally cartesian closed, thus giving rise to a non-split model of dependent type theory.

In the following section, 3.6, we briefly review some of the problems of modelling dependent type theory and then show in Theorem 3.6.20 how to obtain a *split* model of dependent type theory from any realizability pretripos. This is the main result of this chapter.

Then in Section 3.7 we show how to define a category of modest sets over any realizability pretripos and that it also provides a model of dependent type theory. Our account of these models of dependent type theory is a general uniform account in the sense that the already mentioned model of dependent type theory in **Equ** [BBS98] and also models based on assemblies over partial combinatory algebras (see, *e.g.*, [LM91, Jac99]) are special instances. We further relate the category of modest sets to the category of assemblies.

As already mentioned, our approach in this chapter is inspired by the tripos-theoretic approach to realizability over partial combinatory algebras [HJP80, Pit81]. In our joint paper with Aurelio Carboni, Pino Rosolini and Dana Scott [BCRS98], see also [CR99], we have developed a complementary approach to the general notion of realizability for type theory. The complementary approach is based on the theory of exact categories and exact completions, generalizing the exact-completion approach to realizability over partial combinatory algebras. In the theory of realizability over partial combinatory algebras, it has been very useful to have complementary viewpoints, and we believe the same holds for our general notion of realizability for type theory. In Section 3.8 we relate our approach in this chapter to our approach in [BCRS98]. One advantage of the approach in this chapter is that it easily facilitates the description of *split* models of dependent type theory.

Finally, in Section 3.9 I discuss some other closely related work of Abramsky [Abr95], Lambek [Lam94], and Longley [Lon99].

Throughout the chapter we shall see that we leave many interesting questions open. (In particular, we do not undertake a thorough study of a suitable 2-category of WCPC-categories, which would generalize Longley's 2-category of partial combinatory algebras and applicative transformations.) In Section 3.10 we suggest a number of questions for future work.

In this chapter we focus on the general notion of realizability and how it can be used to model dependent type theory; in the following chapter we show how the general notion of realizability can also be used to model logics over the dependent type theory. In Appendix A, I have worked out in very concrete terms a particularly interesting example, namely the dependent type theory and predicate logic for **Equ**. It is basically straightforward to do so using the theorems proved in this and the following chapter. I have nevertheless chosen to include this treatment for the following reasons. First, I hope it may make the abstract treatment in this and the following chapter more accessible to readers not thoroughly familiar with [Jac99]. Indeed it may be helpful to read the appendix in parallel with the treatment in this and the following chapter. Second, we note that when one wants to *use* the type theory and logic to construct objects or prove properties about the model, one often needs to know what the interpretation is in concrete terms and so it makes sense to actually work it out.

3.1 A General Notion of Realizability

In this section we define the notion of a *weakly closed partial cartesian category* (WCPC-category). We shall show how such a category can be seen as a general universe of realizers. In particular, in Section 3.1.2 we show that any partial combinatory algebra gives rise to a weakly closed partial cartesian category. We review the definition of a partial combinatory algebra in Section 3.1.1.

Our notion of a WCPC-category is just a weak version of a cartesian closed category of partial maps. We thus begin by recalling the definition of a partial cartesian category from [RR88]. The wording "partial cartesian" is a shortening of "partial maps on a cartesian category." There is an excellent overview of categories of partial maps in [RR88] to which we refer the reader for much more information on categories of partial maps and their history than we can include here. See also [Ros86].

The following definitions (3.1.1–3.1.7) are from [RR88, Page 101].

Definition 3.1.1. A *p*-**category** is a category $\mathbb{C}$ endowed with a bifunctor $\times\colon \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ which is called **product**, a natural transformation $\Delta\colon (-) \to$

$(-\times-)$ which is called the **diagonal** and two families of natural transformations $\{\, p_{-,Y}\colon (-\times Y) \to (-) \mid Y \in \mathbb{C}\}$ and $\{\, q_{X,-}\colon (X\times -) \to (-) \mid X \in \mathbb{C}\}$ which are called **projections**, satisfying the identities

$$\begin{array}{ll} p_{X,X}\Delta_X = id_X = q_{X,X}\Delta_X & (p_{X,Y} \times q_{X,Y})\Delta_{X\times Y} = id_{X\times Y} \\ p_{X,Y}(id_X \times p_{Y,Z}) = p_{X,Y\times Z} & p_{X,Z}(id_X \times q_{Y,Z}) = p_{X,Y\times Z} \\ q_{X,Y}(p_{X,Y} \times id_Z) = q_{X\times Y,Z} & q_{X,Z}(q_{X,Y} \times id_Z) = q_{X\times Y,Z}. \end{array}$$

Finally we require that the associativity and commutativity isomorphisms α and τ defined as below by

$$\begin{aligned} \alpha_{X,Y,Z} &= ((id_X \times p_{Y,Z}) \times q_{Y,Z}q_{X,Y\times Z})\Delta_{X\times(Y\times Z)} \\ &: X \times (Y \times Z) \to (X \times Y) \times Z \end{aligned}$$

and

$$\tau_{X,Y} = (q_{X,Y} \times p_{X,Y})\Delta_{X\times Y} : X \times Y \to Y \times X$$

are natural in all variables (the components need not be so).

A p-functor $F\colon \mathbb{C} \to \mathbb{D}$ between p-categories is a functor preserving products, projections, and diagonals up to a natural isomorphism.

Definition 3.1.2. Given a map $f\colon X \to Y$ in the p-category $\mathbb{C}$, the **domain** $\operatorname{dom} f\colon X \to Y$ of f is the composite map $p_{X,Y}(id_X \times f)\Delta_X : X \to X$.

Definition 3.1.3. A map $f\colon X \to Y$ in a p-category $\mathbb{C}$ is **total** if $\operatorname{dom} f = id$. We denote the subcategory of total maps $\mathbb{C}_t$.

Example 3.1.4. Let **Ptl** denote the category of sets and partial functions. It is a p-category with total category the usual category **Set** of sets and total functions.

For all objects X and Y in a p-category $\mathbb{C}$, the maps id_X, $p_{X,Y}$, $q_{X,Y}$, and Δ_X are total. If morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ are both total, then their composite $gf\colon X \to Z$ is total. The subcategory $\mathbb{C}_t$ of total maps has binary cartesian products. A p-functor maps total maps to total maps.

Definition 3.1.5. A p-category $\mathbb{C}$ is said to be a **partial cartesian category** if the subcategory $\mathbb{C}_t$ of total maps has a terminal object (and so has finite products).

A partial cartesian category $\mathbb{C}$ is also called a p-category with a **one-element** object.

Remark 3.1.6. Let $\mathbb{C}$ be a partial cartesian category. If on every hom-set $\mathbb{C}(X, Y)$, the extension order is defined as

$$\varphi \leq \psi \iff \varphi = \psi \circ \operatorname{dom} \varphi,$$

then this defines a structure of a bicategory of partial maps on $\mathbb{C}$ [RR88, Proposition 3.4]. See *loc. cit.* and [Car87] for more on bicategories of partial maps.

Definition 3.1.7. Let $\mathbb{C}$ be a partial cartesian category. Then $\mathbb{C}$ is is said to be **closed** if, for every object X in $\mathbb{C}$, there is an adjunction

$$\begin{tikzcd} \mathbb{C}_t \arrow[r, shift left=1.5, "(X\times -)"] \arrow[r, phantom, "\perp"] & \mathbb{C}, \arrow[l, shift left=1.5, "{[X \rightharpoonup -]}"] \end{tikzcd}$$

that is, for all Y and Z in $\mathbb{C}$, there is a natural isomorphism

$$\mathbb{C}(X \times Y, Z) \cong \mathbb{C}_t(Y, [X \rightharpoonup Z]).$$

The object $[X \rightharpoonup Z]$ is referred to as the **partial exponential** of X and Z.

In elementary terms, the above definition says that $\mathbb{C}$ is closed if, for all X and Z in $\mathbb{C}$, there exists an object $[X \rightharpoonup Z]$ and a morphism $\epsilon\colon [X \rightharpoonup Z] \times X \to Z$ in $\mathbb{C}$ such that, for all objects Y, all morphisms $f\colon Y \times X \to Z$ in $\mathbb{C}$, there exists a unique map $f'\colon Y \to [X \rightharpoonup Z]$ in $\mathbb{C}_t$ (note that f' is total!), such that the following diagram commutes in $\mathbb{C}$:

$$\begin{tikzcd} {[X \rightharpoonup Z] \times X} \arrow[r, "\epsilon"] & Z \\ Y \times X \arrow[u, "f' \times id_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

Let $\mathbb{C}$ be a partial cartesian category. Then we say that $\mathbb{C}$ is *weakly closed* if $\mathbb{C}$ satisfies the definition of being closed *except* that the required morphism f' is only required to exist, not to be unique. Explicitly:

Definition 3.1.8. Let $\mathbb{C}$ be a partial cartesian category. Then we say that $\mathbb{C}$ is **weakly closed** if, for all X and Z in $\mathbb{C}$, there exists an object $[X \rightharpoonup Z]$ and a morphism $\epsilon\colon [X \rightharpoonup Z] \times X \to Z$ in $\mathbb{C}$ such that, for all objects Y, all morphisms $f\colon Y \times X \to Z$ in $\mathbb{C}$, there exists a (not necessarily unique) map $f' \colon Y \to [X \rightharpoonup Z]$ in $\mathbb{C}_t$ (note that f' is total!), such that the following diagram commutes in $\mathbb{C}$:

$$
\begin{tikzcd}
{[X \rightharpoonup Z] \times X} \arrow[r, "e"] & Z \\
Y \times X \arrow[u, "f' \times id_X"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

We refer to $([X \rightharpoonup Z], e)$ as the **weak partial exponential of X and Z.**

For simplicity we refer to a weakly closed partial cartesian category as a WCPC-category.

Example 3.1.9. The category **Ptl** of sets and partial functions is a WCPC-category. (In fact, it is not only weakly closed but closed.)

Definition 3.1.10. A **WCPC-functor** is just a p-functor.

Note that a p-functor preserves products and domains and thus maps total morphisms to total morphisms.

We may think of a WCPC-category $\mathbb{C}$ as a general universe of realizers as follows. The category $\mathbb{C}$ is a p-category because realizers may be only partially defined. Intuitively, $[X \rightharpoonup Z]$ is a set of realizers of functions from X to Z. There may be more than one realizer for each function from X to Z, hence f' is only required to exist, not to be unique.

Example 3.1.11. Observe that, trivially, any cartesian closed category is a WCPC-category. Recall that the category **ALat** of algebraic lattices and Scott continuous functions is cartesian closed [DP90, GHK+80]; hence **ALat** is a WCPC-category.

Next, we will show that any partial combinatory algebra gives rise to a WCPC-category, that is *not* necessarily closed.

3.1.1 Partial Combinatory Algebras

We recall the definition of a partial combinatory algebra and present a couple of examples; readers familiar with partial combinatory algebras can skip this section. More detailed treatments of partial combinatory algebras can, for example, be found in [Lon94] or in [Bee85].

A **partial combinatory algebra** (PCA) consists of a set A together with a partial application function $\cdot : A \times A \rightharpoonup A$ and two distinct elements $\mathbf{K}, \mathbf{S} \in A$ satisfying, for all x, y, and z in A,

$$\mathbf{K}x\downarrow, \quad \mathbf{S}x\downarrow, \quad \mathbf{S}xy\downarrow, \qquad \text{and} \qquad \mathbf{K}xy \simeq x, \quad \mathbf{S}xyz \simeq xz(yz),$$

where $P\downarrow$ means that P is defined and where **Kleene equality** $P \simeq Q$ means that P is defined if and only if Q is defined, and in that case they are equal.[1] As above, we often write xy for $x \cdot y$. Letting $\mathbf{I} = \mathbf{SKK} \in A$ one has that $\mathbf{I} \cdot a = a$, for all $a \in A$. A **total combinatory algebra** is just a PCA in which application is total (*i.e.*, $ab\downarrow$, for all $a, b \in A$).

We often simply write A for a PCA $(A, \cdot, \mathbf{K}, \mathbf{S})$. In such a PCA A one has combinatory completeness: for every polynomial term $M(x_1, \ldots, x_n)$ built from variables $x_1, \ldots, x_n$, constants $\underline{c}$ for $c \in A$, and application $\cdot$, there is an element $a \in A$ such that for all $b_1, \ldots b_n \in A$,

$$ab_1 \cdots b_{n-1}\downarrow \qquad \text{and} \qquad ab_1 \cdots b_n \simeq [\![M]\!](b_1, \ldots, b_n),$$

where $[\![M]\!]$ is the partial function $A^n \rightharpoonup A$ obtained by interpreting the polynomial M. To prove this one uses Schönfinkels abstraction rules to define functional abstraction:

$$\begin{aligned} \lambda x.\, x &= \mathbf{I} = \mathbf{SKK} \\ \lambda x.\, M &= \mathbf{K}M && \text{if } x \text{ is not free in } M \\ \lambda x.\, MN &= \mathbf{S}(\lambda x.\, M)(\lambda x.\, N). \end{aligned}$$

Combinatory completeness is then obtained by taking $a = \lambda x_1 \cdots x_n.M$.

Note also the pairing in PCAs, as in the untyped lambda calculus:

$$\langle x, y\rangle = \lambda z.\, zxy = \mathbf{S}(\mathbf{SI}(\mathbf{K}x))(\mathbf{K}y)$$

with projections

$$\pi x = x\mathbf{K} \qquad \text{and} \qquad \pi' x = x(\mathbf{KI}).$$

Then $\langle x, y\rangle\downarrow$ and $\pi\langle x, y\rangle = x$, $\pi'\langle x, y\rangle = y$ (but *not* $\langle \pi x, \pi' x\rangle = x$ — in general no *surjective* pairing function can be encoded [Bar85, Page 134]). Finite sequences can also be encoded as in the untyped lambda calculus (see, *e.g.*, [Bar85, Page 134]). We write

$$[x_1, x_2, \ldots, x_n]$$

for an encoding of the finite sequence of $x_1, x_2, \ldots, x_n \in A$ and we write π_i for an encoding of the i'th projection function.

[1] In other treatments of partial combinatory algebras one will often find that $\mathbf{K}$ and $\mathbf{S}$ are not part of the structure of a partial combinatory algebra, but are just required to exist; see, *e.g.*, [Bee85].

Let $(A, \cdot, \mathbf{K}, \mathbf{S})$ be a PCA. We say that a subset $B \subseteq A$ is a **sub partial-combinatory-algebra of** A (sub-PCA) if and only if B contains $\mathbf{K}$ and $\mathbf{S}$ and is closed under partial application. Note that if B is a sub-PCA of $(A, \cdot, \mathbf{K}, \mathbf{S})$ then any element defined using lambda calculus and constants from B is again an element of B.

Convention 3.1.12. For sets $X, Y \subseteq A$ we write $X \supset Y$ for the set

$$\{ f \in A \mid \forall a \in X.\ f \cdot a \downarrow \text{ and } f \cdot a \in Y \}.$$

We sometimes write $f\colon X \supset Y$ for $f \in (X \supset Y)$.

Example 3.1.13. Consider the set $\mathbb{N}$ of natural numbers equipped with **Kleene application:** $m \cdot n \simeq \{m\}(n)$, where $\{m\}$ denotes the partial recursive function coded by m. The existence of $\mathbf{K}$ and $\mathbf{S}$ with the required properties is an immediate consequence of the S-M-N theorem. We refer to this PCA as $\mathcal{K}_1$, or **Kleene's first model.**[2]

Example 3.1.14. Let Λ denote the set of terms of the untyped lambda calculus over a countably infinite set of variables. Let Λ/β be its quotient modulo β-equality (see, *e.g.*, [Bar85]) with the induced application. This defines a total combinatory algebra: for $\mathbf{K}$ and $\mathbf{S}$ we may take the equivalence classes of the lambda terms $\lambda x, y.\, x$ and $\lambda x, y, z.\, xz(yz)$.

Example 3.1.15. Let $\mathbb{N}$ denote the set of natural numbers and let $\mathbb{P} = P\mathbb{N}$ be its powerset. Suppose that $\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a coding function for pairs and that $[-, \cdots, -] : P_{fin}\,\mathbb{N} \to \mathbb{N}$ is a coding function for finite subsets of $\mathbb{N}$. Recall that $\mathbb{P}$ can be viewed as a topological space with the Scott topology and that a function $f : \mathbb{P} \to \mathbb{P}$ then is **continuous** if and only if f preserves inclusion and $f(\bigcup A_i) = \bigcup(fA_i)$ whenever $A_0 \subseteq A_1 \subseteq \cdots \subseteq \mathbb{P}$. For a continuous function $f : \mathbb{P} \to \mathbb{P}$, we define the **graph** of f to be the set

$$\mathrm{Graph}(f) = \{ \langle [x_1, \ldots, x_n], y \rangle \mid y \in f(\{x_1, \ldots, x_n\}) \} \in \mathbb{P}.$$

Since f is continuous, it is completely determined by its graph: for any $A \in \mathbb{P}$, $f(A) = \{ y \mid \exists x_1, \ldots, x_n \in A.\ \langle [x_1, \ldots, x_n], y \rangle \in \mathrm{Graph}(f) \}$. The operation

$$A \cdot B = \{ y \mid \exists x_1, \ldots, x_n \in B.\ \langle [x_1, \ldots, x_n], y \rangle \in A \}$$

is a continuous application operation $\cdot : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$, and $(\mathbb{P}, \cdot)$ forms a total combinatory algebra, called the **graph model.**

[2] The definition of $\mathcal{K}_1$ is dependent on a particular encoding of the partial recursive functions as natural numbers; however, for most purposes, the choice of encoding will be irrelevant.

Example 3.1.16. Let RE denote the recursively enumerable (r.e.) subsets of $\mathbb{N}$. If in the previous example we take the coding functions $\langle -,- \rangle$ and $[-,\cdots,-]$ to be *recursive*, then $(RE,\cdot)$ forms a sub-PCA of $(\mathbb{P},\cdot)$ as, clearly, **K** and **S** may be chosen to be r.e. subsets. We refer to RE as the **recursively enumerable graph model.**

There are many more examples of partial combinatory algebras (see, for example, [Lon94]).

3.1.2 Constructing a WCPC-category from a Partial Combinatory Algebra

We now show how any partial combinatory algebra in a direct way gives rise to a WCPC-category. Let $(A,\cdot)$ be a PCA. By an **A-definable (total) function** we mean a function f from A to A for which there exists an element $a \in A$ such that, for all $b \in A$, $a \cdot b \downarrow$ and $a \cdot b = f(b)$. The set of A-definable functions form a monoid under composition; we refer to this monoid as the **monoid of A-definable functions.** By an **A-definable partial function** we mean a partial function f from A to A for which there exists an element $a \in A$ such that, for all $b \in A$, we have that $a \cdot b \simeq f(b)$.

Definition 3.1.17. Let $(A,\cdot)$ be a PCA.

1. We define the category $\mathbb{C}(A)_t$ to be the idempotent splitting of the monoid of A-definable total functions. That is, objects are total A-definable idempotents and a morphism $f\colon X \to Y$ is an A-definable total function such that $YfX = f$. The identity on X is X itself and composition is composition of total functions.

2. We define the **category $\mathbb{C}(A)$ induced by A** to be the category with objects total A-definable idempotents (*i.e.*, same objects as $\mathbb{C}(A)_t$) and with morphisms $f\colon X \to Y$ A-definable partial functions f satisfying that $YfX = f$ (as partial functions). The identity on X is X and composition is composition of partial functions.

Of course, this definition is related to and inspired by the work of Dana Scott [Sco80] and the treatment of C-monoids in [LS86]. We do not pause here to study what universal property $\mathbb{C}(A)$ may have, but content ourselves with showing that $\mathbb{C}(A)$ indeed is a WCPC-category:

Proposition 3.1.18. *Let A be a PCA. Then the category $\mathbb{C}(A)$ induced by A is a WCPC-category and its subcategory of total maps is equivalent to $\mathbb{C}(A)_t$.*

Proof. The bifunctor $\times : \mathbb{C}(A) \to \mathbb{C}(A)$ is defined on objects X and Y by

$$X \times Y = \lambda z.\ \langle X(\pi z), Y(\pi' z)\rangle$$

and on morphisms $f : X \to X'$ and $g : Y \to Y'$ by

$$f \times g = \lambda z.\ \langle f(\pi z), g(\pi' z)\rangle.$$

The diagonal natural transformation Δ is given by

$$\Delta_X : X \to X \times X = \lambda x.\ \langle Xx, Xx\rangle$$

and the projections are given by

$$\begin{aligned} p_{X,Y} &: X \times Y \to X = \pi \circ (X \times Y) \\ q_{X,Y} &: X \times Y \to Y = \pi' \circ (X \times Y) \end{aligned}$$

It can then be verified that the induced isomorphisms $\alpha_{X,Y,Z}$ and $\tau_{X,Y}$ indeed are natural in all variables, so that $\mathbb{C}(A)$ indeed is a p-category.

By definition $f : X \to Y$ is total in the sense of a p-category if $p_{X,Y}(id_X \times f)\Delta_X : X \to X$ equals the identity on X. Note that, $f : X \to Y$ entails that $fX = f$ (as partial functions). Further note that, for all $a \in A$,

$$\begin{aligned} p_{X,Y}(id_X \times f)\Delta_X(a) &= p_{X,Y}(id_X \times f)\langle X(a), X(a)\rangle \\ &= \begin{cases} p_{X,Y}\langle X(a), f(X(a))\rangle & \text{if } f(X(a)) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases} \\ &= \begin{cases} X(a) & \text{if } f(a) \text{ is defined} \\ \text{undefined} & \text{otherwise.} \end{cases} \end{aligned}$$

Since the identity on X is X itself, we clearly have that f is total in the sense of a p-category just in case f is a morphism of $\mathbb{C}(A)_t$.

To show that $\mathbb{C}(A)$ is a partial cartesian category, it just remains to show that $\mathbb{C}(A)_t$ has a terminal object. The terminal object 1 in $\mathbb{C}(A)_t$ is the idempotent $\lambda x.\, \mathbf{K}$.

It now only remains to show that $\mathbb{C}(A)$ is weakly closed. For objects X and Y in $\mathbb{C}(A)$ we define $[X \rightharpoonup Z]$ to be the (total) idempotent

$$\lambda f.\, \lambda x.\, Z(f(X(x))).$$

(Note that $[X \rightharpoonup Z]$ of course is A-definable, by the element

$$\lambda a.\, \lambda b.\, a_Z(a(a_X(b))) \in A,$$

where a_X and a_Z witness the definability of X and Z.) The morphism $e\colon [X \rightharpoonup Z] \times X \to Z$ is defined to be the (partial) function $\lambda u.\ (\pi u)(\pi' u)$. Let $f\colon Y \times X \to Z$ be any morphism in $\mathbb{C}(A)$ and define $f'\colon Y \to [X \rightharpoonup Z]$ to be $\lambda y. [X \rightharpoonup Z](\lambda x. f\langle Yy, x\rangle)$. Note that f' is total. One can then verify that the diagram

$$\begin{tikzcd}
{[X \rightharpoonup Z] \times X} \arrow[r, "e"] & Z \\
Y \times X \arrow[u, "f' \times id_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes in $\mathbb{C}(A)$, as required. □

Convention 3.1.19. We will often refer to the category $\mathbb{C}(A)$ induced by a PCA A as **the WCPC-category induced by** A, thus implicitly referring to the above proposition.

For a PCAA, we define a morphism $U\colon \mathbb{C}(A) \to \mathbf{Ptl}$ of WCPC-categories as follows. On an object $X \in \mathbb{C}(A)$, we set

$$U(X) = \{\, X(a) \mid a \in A \,\},$$

and for a morphism $f\colon X \to Y$ in $\mathbb{C}(A)$, we set

$$U(f) = f.$$

Note that U indeed is a WCPC-functor; in particular U maps $\mathbb{C}(A)_t$ into **Set** and U applied to the terminal object in $\mathbb{C}(A)_t$ is a terminal object in **Set**.

3.2 Regular Categories and Regular Fibrations

In Section 3.2.1 we recall the definition of a regular category and some basic properties of regular categories; the material is entirely standard and may be skipped if you are familiar with regular categories. In Section 3.2.2 we recall the definition of a regular fibration from [Jac99]. Readers familiar with [Jac99] may skip this section.

3.2.1 Regular Categories

We recall the definition of a regular category and some basic properties. See, for example, [BGvO71, FS90, Bor94b] for more background on regular categories. First recall that a **regular epimorphism** is an epimorphism which occurs as a coequalizer, *i.e.*, $e\colon Y \to Z$ is a regular epimorphism if there exists a pair of morphisms $f, g\colon X \to Y$ such that

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "e"] & Z \end{tikzcd}$$

is a coequalizer diagram. Also recall that a category is said to be **left exact**, or **lex** for short, if it has all finite limits: equalizers, pullbacks, *etc.*. Also recall that a kernel pair (f_0, f_1) of an arrow $f\colon X \to Y$ is the pullback of f with itself, as in the following diagram:

$$\begin{tikzcd} K \arrow[r, "f_0"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Finally, recall that, in a category with pullbacks, a regular epimorphism is the coequalizer of its kernel pair.

Definition 3.2.1. A category $\mathbb{C}$ is **regular** when

1. $\mathbb{C}$ has finite limits;
2. every kernel pair has a coequalizer;
3. regular epimorphisms are stable under pullbacks (*i.e.*, the pullback of a regular epimorphism along any morphisms is again a regular epimorphism).

It follows that a regular category is the same as a left exact category with a (regular epi, mono) stable factorization system [FK72]. Indeed, the image of a morphism f is obtained as the coequalizer of the kernel pair of f.

A functor $F\colon \mathbb{C} \to \mathbb{D}$ is **exact** (also called **regular**) if it preserves finite limits and coeqalizers of kernel pairs (or, equivalently, preserves finite limits and regular epis). Note that we use the term "exact" in the sense of Barr [BGvO71]; in particular, it should not be confused with exact in the sense of preserving finite limits and finite colimits.

3.2.2 Regular Fibrations

Recall from [Jac99, Definition 4.2.1, Page 233] that a **regular fibration** $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ is a fibration which

1. is a fibred preorder;
2. has finite products in its base category $\mathbb{B}$;
3. has fibred finite products (for $\top$ and $\wedge$);
4. has fibred equality $(Eq_{I,J} \dashv \delta(I,J)^*)$ satisfying Frobenius (for $=$);
5. has simple coproducts $(\coprod_{(I,J)} \dashv \pi^*_{I,J})$ satisfying Frobenius (for $\exists$).

Remark 3.2.2. A category $\mathbb{B}$ is regular if and only if $\begin{array}{c}\mathrm{Sub}(\mathbb{B})\\ \downarrow\\ \mathbb{B}\end{array}$ is a regular fibration [Jac99, Theorem 4.4.4]. In particular, if $\mathbb{B}$ is a topos, then the subobject fibration on $\mathbb{B}$ is a regular fibration which is equivalent, as a regular fibration, to the fibration obtained from the split indexed category $\mathbb{B}(-, \Omega_{\mathbb{B}})$, where $\Omega_{\mathbb{B}}$ is the subobject classifier of $\mathbb{B}$.

A regular fibration models **regular logic**, the fragment of first-order (intuitionistic) logic using only $=$, $\wedge$, $\top$, and $\exists$.

3.3 Assemblies over Regular Fibrations

For any regular fibration, we may define a category of assemblies over it as follows. For particular regular fibrations, this construction specializes to assemblies over algebraic lattices as in [BBS98] and assemblies over partial combinatory algebras [CFS88, FS87, Car95, Lon94], see Examples 3.6.10 and 3.6.11 in Section 3.6.2. The definition below is phrased using the internal language of a regular fibration; in the remark following the definition we recall what those logical definitions mean in categorical terms.

Definition 3.3.1. Let $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ be a regular fibration. Define $\mathbf{Asm}(p)$ to be the category with

objects pairs (X, E_X) with $X \in \mathbb{B}$, $E_X \in \mathbb{E}_X$, satisfying that, for all global elements $c_X : 1 \to X$ in $\mathbb{B}$,

$$\emptyset \mid \emptyset \vdash E_X(c_X)$$

is valid in the logic of p.

morphisms from (X, E_X) to (Y, E_Y) are morphisms $f : X \to Y$ in $\mathbb{B}$ for which

$$x : X \mid E_X(x) \vdash E_Y(f(x))$$

is valid in the logic of p.

Identities and composition are as in $\mathbb{B}$. That is, the identity morphism on (X, E_X) is the identity on X and the composition of two morphisms is the composition of the morphisms in $\mathbb{B}$.

Remark 3.3.2. The condition on objects in the above definition can be expressed categorically as follows. For all global elements $c_X : 1 \to X$ in $\mathbb{B}$, $\top_1 \leq_1 c_X^*(E_X)$ in the fibre $\mathbb{E}_1$ over 1. The condition on morphisms $f : (X, E_X) \to (Y, E_Y)$ can be expressed categorically as follows: we require that $E_X \leq f^*(E_Y)$ in the fibre $\mathbb{E}_X$ over X.

Proposition 3.3.3. *Let* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *be a regular fibration. If* $\mathbb{B}$ *is regular,* 1 *is regular projective in* $\mathbb{B}$*, and* p *has coproducts satisfying the Frobenius property, then* $\mathbf{Asm}(p)$ *is a regular category.*

Proof. The terminal object is $(1_{\mathbb{B}}, T_{1_{\mathbb{B}}})$.

The product of (X, E_X) and (Y, E_Y) is $(X \times Y, E)$ where

$$(x, y) : X \times Y \vdash E(x, y) \stackrel{\text{def}}{=} E_X(x) \wedge E_Y(y)$$

with projections π and π' the projections of $X \times Y$. (Categorically, E is $\pi^*(E_X) \wedge \pi'^*(E_Y)$.)

The equalizer of $f, g : (X, E_X) \to (Y, E_Y)$ is (Z, E_Z), where

$$Z \xrightarrow{m} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

is an equalizer diagram in $\mathbb{B}$, and E_Z is defined as follows:

$$z : Z \vdash E_Z(z) \stackrel{\text{def}}{=} E_X(m(z)).$$

Categorically, $E_Z = m^* E_X$.

It remains to show that $\mathbf{Asm}(p)$ has stable images. So suppose

$$f : (X, E_X) \to (Y, E_Y)$$

in $\mathbf{Asm}(p)$. Then $f: X \to Y$ in $\mathbb{B}$ and we can consider the image factorization of f in $\mathbb{B}$ (since $\mathbb{B}$ is regular):

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & Y \\ & \mathrm{Im}(f) \arrow[ur, tail, "m"'] & \end{tikzcd}$$

Let $W = \mathrm{Im}(f)$ and let

$$w: W \vdash E_W(w) \stackrel{\text{def}}{=} \exists x: X.\ e(x) = w \wedge E_X(x).$$

Categorically, $E_W = \coprod_e E_X$. We then claim that

1. (W, E_W) is an object of $\mathbf{Asm}(p)$;

2. $$\begin{tikzcd} (X, E_X) \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & (Y, E_Y) \\ & (W, E_W) \arrow[ur, tail, "m"'] & \end{tikzcd}$$

   forms an image factorization in $\mathbf{Asm}(p)$;

3. images as in item 2 are stable under pullback.

Ad 1: Let $c_W: 1 \to W$ be any global element. By definition of E_W we are to show that

$$\emptyset \mid \emptyset \vdash \exists x: X.\ e(x) = c_W \wedge E_X(x)$$

is valid in the logic of p. Since $e: X \twoheadrightarrow W$ is an epi and 1 is regular projective in $\mathbb{B}$, there is a morphism $x: 1 \to X$ in $\mathbb{B}$ such that $ex = c_w$. By the assumption that $(X, E_X) \in \mathbf{Asm}(p)$, we have the required. (We here use that external existence implies internal existence [Jac99, Page 255] and that external equality implies internal equality.)

Ad 2: We first verify that e and m are indeed morphisms of $\mathbf{Asm}(p)$. For e we are to show that

$$x: X \mid E_X(x) \vdash E_W(e(x)) \tag{3.1}$$

is valid in the logic of p. By definition of E_W, we have that (3.1) is equivalent to

$$x : X \mid E_X(x) \vdash \exists x : X.\ e(x) = e(x) \wedge E_X(x),$$

which is clearly valid. For m we are to show that

$$y : W \mid E_W(y) \vdash E_Y(m(y))$$

is valid in the logic of p, that is, that

$$y : W \mid \exists x : X.\ e(x) = y \wedge E_X(x) \vdash E_Y(m(y)) \tag{3.2}$$

is valid in the logic of p. But (3.2) is clearly valid because, reasoning internally, under the given assumption, $m(y) = m(e(x)) = f(x)$ (the latter because externally we have $f = me$); f is an arrow in $\mathbf{Asm}(p)$; and $E_X(x)$ holds by assumption.

We now verify that $m \circ e$ is indeed an image factorization of f. So suppose that f also factors as $h \circ g$, as in the diagram

$$\begin{tikzcd}
(X, E_X) \arrow[rr, "f"] \arrow[dr, two heads, "e"] \arrow[ddr, two heads, "g"'] & & (Y, E_Y) \\
& (W, E_W) \arrow[ur, tail, "m"'] & \\
& (Z, E_Z) \arrow[uur, tail, "h"'] &
\end{tikzcd}$$

Then $h \circ g$ is also a factorization of f in $\mathbb{B}$; hence there exists a unique $u : W \to Z$ in $\mathbb{B}$ such that

$$u \circ e = g \qquad \text{and} \qquad h \circ u = m$$

in $\mathbb{B}$. Thus it suffices to show that u is an arrow in $\mathbf{Asm}(p)$, *i.e.*, to show that

$$y : W \mid E_W(y) \vdash E_Z(u(y))$$

is valid in the logic of p. We show this by arguing in the logic of p: Suppose that $E_W(y)$, *i.e.*, that $\exists x : X.\ e(x) = y \wedge E_X(x)$. Then $u(y) = g(x)$ and thus, as $E_X(x)$ and g is an arrow in $\mathbf{Asm}(p)$, also $E_Z(g(x)) = E_Z(u(y))$, as required.

Ad 3: Suppose that

$$\begin{tikzcd}
(P, E_P) \arrow[r, "h'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & (X, E_X) \arrow[d, "f"] \\
(Z, E_Z) \arrow[r, "g"'] & (Y, E_Y)
\end{tikzcd}$$

is a pullback in $\mathbf{Asm}(p)$. We are to show that the bottom square in

$$\begin{tikzcd}
(P, E_P) \arrow[r, "h'"] \arrow[d, two heads, "e'"'] & (X, E_X) \arrow[d, two heads, "e"'] \\
(U, E_U) \arrow[r, dashed, "h"] \arrow[d, tail, "m'"'] & (W, E_W) \arrow[d, tail, "m"] \\
(Z, E_Z) \arrow[r, "g"'] & (Y, E_Y)
\end{tikzcd}$$

is a pullback (where (U, E_U) is the image of f' and (W, E_W) is the image of f). By stability of image factorizations in $\mathbb{B}$ we have that both squares in the diagram

$$\begin{tikzcd}
P \arrow[r, "h'"] \arrow[d, two heads, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "e"'] \\
U \arrow[r, dashed, "h"] \arrow[d, tail, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, tail, "m"] \\
Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

are pullbacks in $\mathbb{B}$. Thus it suffices to show that

$$E_u \stackrel{\mathrm{def}}{=} \textstyle\coprod_{e'} E_P$$

is isomorphic to

$$(m')^* E_Z \wedge h^* E_W.$$

But this is easy:

$$\begin{aligned}
E_u &= \textstyle\coprod_{e'} E_p \\
&= \textstyle\coprod_{e'}((f')^* E_Z \wedge (h')^* E_X) \\
&\cong \textstyle\coprod_{e'}((e')^*(m')^* E_Z \wedge (h')^* E_X) \\
&\cong (m')^* E_Z \wedge \textstyle\coprod_{e'}(h')^* E_X) && \text{by Frobenius} \\
&\cong (m')^* E_Z \wedge h^* \textstyle\coprod_{e} E_X && \text{by Beck-Chevalley} \\
&= (m')^* E_Z \wedge h^* E_W.
\end{aligned}$$

(Note the use of the Frobenius and Beck-Chevalley conditions; they explains the assumption in the proposition regarding coproducts along all maps in $\mathbb{B}$.) □

For assemblies constructed over a regular fibration we may define functors to and from the base category as follows, generalizing the case of assemblies over a partial combinatory algebra (see, *e.g.*, [Lon94]).

Definition 3.3.4. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a regular fibration. Define the functor $\nabla\colon \mathbb{B} \to \mathbf{Asm}(p)$ as follows:

$$\nabla(X) = (X, \top_X) \qquad \text{and} \qquad \nabla(f\colon X \to Y) = f.$$

Further, let $\Gamma\colon \mathbf{Asm}(p) \to \mathbb{B}$ be the functor defined as follows:

$$\Gamma(X, E_X) = X \qquad \text{and} \qquad \Gamma(f) = f.$$

Note that ∇ is clearly full and faithful and that Γ is faithful.

Proposition 3.3.5. *Let* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{smallmatrix}$ *be a regular fibration. The functor*

$$\Gamma\colon \mathbf{Asm}(p) \to \mathbb{B}$$

is left adjoint to ∇*:*

$$\mathbf{Asm}(p) \underset{\nabla}{\overset{\Gamma}{\underset{\perp}{\rightleftarrows}}} \mathbb{B}.$$

Proof. Clearly,

$$\frac{\Gamma(X, E_X) = X \longrightarrow Y}{\overline{(X, E_X) \longrightarrow (Y, \top_Y) = \nabla(Y)}}$$

because, for any $f\colon X \to Y$ in $\mathbb{B}$, we trivially have that $x\colon X \mid E_X(x) \vdash \top_Y(f(x))$ is valid in p. □

Proposition 3.3.6. *Let* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *be a regular fibration satisfying the condition os Proposition 3.3.3 so that* $\mathbf{Asm}(p)$ *is regular. Then both* ∇ *and* Γ *are regular functors.*

Proof. The functor ∇ preserves limits as a right adjoint (Proposition 3.3.5). It clearly preserves image factorizations. The functor Γ preserves regular epis as a left adjoint. It clearly preserves finite limits. $\square$

Remark 3.3.7. For a regular fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{smallmatrix}$ over **Set**, the functor

$$\Gamma\colon \mathbf{Asm}(p) \to \mathbf{Set}$$

is naturally isomorphic to the global sections functor $\mathrm{Hom}_{\mathbf{Asm}(p)}(1, -)$.

We leave the question of what universal property the construction of the category of assemblies over a regular fibration satisfies. Some useful ideas for answering this question can be found in Longley's thesis [Lon94], in which Longley gave a universal property for the construction of the category of assemblies over a partial combinatory algebra.

3.4 Pretriposes

Consider a regular fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{smallmatrix}$ over **Set** with coproducts satisfying Frobenius. We now know that the category of assemblies $\mathbf{Asm}(p)$ is regular. In this section we show that by imposing further conditions on $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{smallmatrix}$ besides regularity, the resulting category $\mathbf{Asm}(p)$ is locally cartesian closed.[3] In Section 3.4.1 we show that any WCPC-category gives rise to a fibration over **Set** meeting these conditions. We restrict attention to fibrations with base category **Set** since those suffice for our applications — the interested reader should not have any difficulty with generalizing the development to more general base categories.

Definition 3.4.1. A fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{smallmatrix}$ is called a **pretripos** if

[3] Recall that a category $\mathbb{C}$ is locally cartesian closed if every slice $\mathbb{C}/X$ is cartesian closed. A locally cartesian closed category has finite limits.

1. it is a fibred preorder;

2. it is fibred cartesian closed (for $\top$, $\wedge$, $\supset$);

3. it has coproducts $\exists_u \dashv u^*$ along all maps $u: I \to J$ in the base **Set**;

4. it has products $u^* \dashv \forall_u$ along all maps $u: I \to J$ in the base **Set**.

If, in addition, $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{smallmatrix}$ has fibred finite coproducts (for $\perp$, $\vee$), we say that the fibration is a **pretripos with disjunction**.

Recall from Chapter 2 that it is part of the definition of having coproducts and products that the Beck-Chevalley condition holds. A pretripos (with disjunction) is said to be **split** if it is a split fibration, reindexing preserves the (bi)cartesian closed structure on the nose, and the Beck-Chevalley conditions hold with equality and not only isomorphisms.

Remarks 3.4.2.

(i) The name "pretripos" has been chosen to reflect the fact that such a fibration is weak version of a tripos [HJP80, Pit81] (see also Chapter 5): the essential difference is that a pretripos with disjunction is not required to have a generic object.

(ii) A pretripos $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{smallmatrix}$ is a regular fibration: since we have coproducts along all maps and p is fibred cartesian closed, Frobenius automatically holds for $\exists$ and we have equality satisfying Frobenius [Jac99, Lemma 1.9.12]. Hence if $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{smallmatrix}$ is a pretripos, $\mathbf{Asm}(p)$ is a (well-defined) regular category by Proposition 3.3.3.

(iii) A pretripos with disjunction models predicate logic and is a first-order hyperdoctrine in the sense of Pitts [Pit99] and a first-order fibration in the sense of Jacobs [Jac99].[4] ("First-order" is perhaps a bit misleading

[4] The only difference between a first-order fibration over **Set** in the sense of Jacobs [Jac99] and a pretripos with disjunction is that we require products and coproducts along all morphisms, not only projections. As mentioned in [Jac99, 4.3.7, Page 253], for a first-order fibration one may in fact define left and right adjoints to u^* for all maps $u: I \to J$, but they will not necessarily satisfy the Beck-Chevalley condition. We require explicitly that these adjoints do satisfy the Beck-Chevalley condition. This extra requirement is needed for Proposition 3.3.3, it is met in all our applications and, moreover, it makes the connection with the standard definition of tripos [HJP80] very simple: a tripos is a pretripos with disjunction with a weak generic object, see Chapter 5.

since it only refers to the fact that we cannot quantify over all relations; it is possible to quantify over all types, including higher types, so the logic, which a pretripos models, is more like what is sometimes called λ-logic [AB97]) Thus there are many examples of pretriposes; in particular, any tripos over **Set** is a pretripos with disjunction.

3.4.1 Realizability Pretriposes

Let $\mathbb{C}$ be a WCPC-category and $U\colon \mathbb{C} \to \mathbf{Ptl}$ a WCPC-functor. We now show how to define a pretripos $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ over **Set** from $\mathbb{C}$ and U. We refer to it as the **realizability pretripos over** $\mathbb{C}$, thus omitting explicit mentioning of U (which we think of as a forgetful functor). The idea is as for realizability triposes over partial combinatory algebras [HJP80], the difference being that here the realizers will be drawn from $\mathbb{C}$ instead of from a PCA. By a realizer we shall mean a morphism f in $\mathbb{C}$.

We define a functor $\Psi\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ as follows. For a set I, $\Psi(I)$ is the preorder with objects pairs of the form (A, φ), where A is an object of $\mathbb{C}$ and $\varphi\colon I \to P(UA)$ in **Set**. The less-than-or-equal relation is denoted $\vdash$ and the objects are preordered by decreeing that

$$(A, \varphi) \vdash (B, \psi)$$

if and only if

$$\exists g \in \mathbb{C}(A, B).\ \forall i \in I.\ \forall a \in \varphi(i).\ U(g)(a)\downarrow \text{ and } U(g)(a) \in \psi(i).$$

We refer to the objects of $\Psi(I)$ as **predicates on** I or just as **predicates**. When the object A is clear from context we sometimes just write φ for (A, φ). Further, we sometimes write $(A, \varphi)(i)$ for $\varphi(i)$. We refer to the object A in (A, φ) as the **underlying object of realizers for** φ.

For a morphism $u\colon I \to J$, $\Psi(u)$ is essentially composition:

$$\Psi(u)(A, \varphi\colon J \to P(UA)) = (A, \varphi \circ u\colon I \to P(UA)).$$

Note that $\Psi(u)$ is indeed monotone, hence a well-defined functor. Moreover, Ψ is clearly a functor, *i.e.*, Ψ is a split indexed category. The resulting split fibration (obtained by the Grothendieck construction) is written $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$.

The fibred cartesian closed structure is given as follows. In the fibre over I,

$$\top = (1, i \mapsto U1),$$

where 1 is the terminal object in the category of total maps $\mathbb{C}_t$. The object $\top$ is indeed the terminal object in the fibre over I, because for any other object $(A, \varphi\colon I \to P(UA))$, there exists a map $g\colon A \to 1$ in $\mathbb{C}_t$ which U maps to a total function $U(g)$ so that g realizes $(A,\varphi) \vdash \top$.

For predicates (A,φ) and (B,ψ) over I, define $(A,\varphi) \wedge (B,\psi)$ to be

$$(A \times B, i \mapsto \{\, (a,b) \mid a \in \varphi(i) \text{ and } b \in \psi(i) \,\}).$$

(Note that this definition makes sense because U preserves products.) It is straightforward to see that $\wedge$ so defined gives binary products in the fibre over I.

For exponentials (implication), we define $(A,\varphi) \supset (B,\psi)$ to be

$$\begin{aligned}&([A \rightharpoonup B],\\ &i \mapsto \{\, g \in U[A \rightharpoonup B] \mid \forall a \in \varphi(i).\ U(e)(g,a)\downarrow \text{ and } U(e)(g,a) \in \psi(i) \,\}),\end{aligned}$$

where $([A \rightharpoonup B], e)$ is the weak partial exponential of A and B. (Note that this definition makes sense because U preserves products.) Let us verify the adjunction:

$$\frac{(A,\varphi_1) \wedge (B,\varphi_2) \vdash (C,\varphi_3)}{\overline{(A,\varphi_1) \vdash ((B,\varphi_2) \supset (C,\varphi_3))}} = ([B \rightharpoonup C], \psi).$$

Suppose $(A,\varphi_1) \wedge (B,\varphi_2) \vdash (C,\varphi_3)$ via realizer $h\colon A \times B \to C$. Then there exists an $h'\colon A \to [B \rightharpoonup C]$ in $\mathbb{C}_t$ such that the diagram

$$\begin{tikzcd} {[B \rightharpoonup C] \times B} \arrow[dr, "e"] & \\ A \times B \arrow[u, "h' \times id_B"] \arrow[r, "h"'] & C, \end{tikzcd}$$

commutes, where $([B \rightharpoonup C], e)$ is the weak exponential of B and C. We claim that h' is a realizer for $(A,\varphi_1) \vdash (B,\varphi_2) \supset (C,\varphi_3)$. Indeed, let $i \in I$ be arbitrary and let $a \in \varphi_1(i)$ be arbitrary. Note that $U(h')(a)$ is defined because $U(h')$ is total since h' is so. It remains to show that $U(h')(a) \in ((B,\varphi_2) \supset (C,\varphi_3))(i)$. To this end, let $b \in \varphi_2(i)$ be arbitrary. Then we have that

$$\begin{aligned} U(e)(U(h')(a),b) &= U(e)(U(h')(a),U(id)b)\\ &= U(e) \circ U(h' \times id)(a,b)\\ &= U(e \circ h' \times id)(a,b)\\ &= U(h)(a,b)\\ &\in \varphi_3(i) \qquad \text{by assumption.} \end{aligned}$$

For the other direction, suppose $h' \colon A \to [B \rightharpoonup C]$ is a realizer for $(A, \varphi_1) \vdash (B, \varphi_2) \supset (C, \varphi_3)$. We then claim that $e \circ \langle h', id_B \rangle \colon A \times B \to C$ is a realizer for $(A, \varphi_1) \wedge (B, \varphi_2) \vdash (C, \varphi_3)$. Indeed, let $i \in I$ and $(a, b) \in ((A, \varphi_1) \wedge (B, \varphi_2))(i)$ be arbitrary. By assumption $U(h')(a) \downarrow$ and $U(h')(a) \in ((B, \varphi_2) \supset (C, \varphi_3))(i)$, so for all $b \in \varphi_2(i)$, we have that $U(e)(U(h')a, b) \downarrow$ and $U(e)(U(h')a, b) = U(e \circ \langle h', id_B \rangle)(a, b) \in \varphi_3(i)$. The required follows.

It is easy to verify that, for any $u \colon I \to J$ in the base category **Set**, the reindexing functor u^* preserves the fibred cartesian closed structure on the nose. In summa, we now have that $\begin{smallmatrix}\mathbf{UFam}(\mathbb{C})\\ \downarrow \\ \mathbf{Set}\end{smallmatrix}$ is *split fibred* cartesian closed.

Let $u \colon I \to J \in \mathbf{Set}$ and suppose (A, φ) is a predicate in the fibre over I. For the coproduct along u we define

$$\exists_u(A, \varphi) = (A, j \mapsto \bigcup_{i \in I} \{\, \varphi(i) \mid u(i) = j \,\}).$$

(Note that in the typical case, where $u = \pi' \colon I \times J \to J$, then $\exists_u(A, \varphi)$ simplifies to $(A, j \mapsto \bigcup_{i \in I} \varphi(i, j))$.)

It is easy to verify that $\exists_u \dashv u^*$. For the Beck-Chevalley condition, suppose that

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "h"'] & K \arrow[d, "v"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

is a pullback in **Set**. Now $\exists_h k^* = u^* \exists_v$ because, using that P is a pullback, we have

$$\begin{aligned} u^* \exists_v(A, \varphi) &= (A, i \mapsto \bigcup_{k' \in K} \{\, \varphi(k') \mid v(k') = u(i) \,\}) \\ &= (A, i \mapsto \bigcup_{k' \in K} \{\, \varphi(k') \mid \exists p \in P.\ h(p) = i \text{ and } k(p) = k' \,\}) \\ &= (A, i \mapsto \bigcup_{p \in P} \{\, \varphi(k(p)) \mid h(p) = i \,\}) \\ &= \exists_h k^*(A, \varphi). \end{aligned}$$

Thus we have split coproducts.

For the product along u we define $\forall_u(A, \varphi)$ to be

$$\left([1 \rightharpoonup A], j \mapsto \bigcap_{i \in I} \left((u(i) =_J j) \supset \varphi(i) \right) \right),$$

where

- $[1 \rightharpoonup A]$ is the weak partial exponential of the terminal object 1 in $\mathbb{C}_t$ and A
- $\big(u(i) =_J j\big) = \bigcup\{\, U1 \mid u(i) = j \,\}$
- $\big((u(i) =_J j) \supset \varphi(i)\big)$ equals

  $$\{\, g \in U[1 \rightharpoonup A] \mid \forall b \in (u(i) =_J j).\ U(e)(g,b)\downarrow \text{ and } U(e)(g,b) \in \varphi(i) \,\}$$

(Note that in the typical case, where $I \neq \emptyset$ and $u = \pi' \colon I \times J \to J$, then $\forall_u(A, \varphi)$ simplifies to $(A, j \mapsto \bigcap_{i \in I} \varphi(i,j))$.)

For the adjointness we are to show

$$\begin{array}{rcl} & \underline{\underline{(B,\psi) \vdash \forall_u(A,\varphi)}} & \text{in } \mathbf{UFam}(\mathbb{C})_J \\ u^*(B,\psi) = & (B, \psi \circ u) \vdash (A,\varphi) & \text{in } \mathbf{UFam}(\mathbb{C})_I \end{array}$$

To this end, suppose first that $h \colon B \to [1 \rightharpoonup A]$ is a realizer for $(B,\psi) \vdash \forall_u(A,\varphi)$. That means that

$$\forall j \in J.\ \forall b \in \psi(j).\ U(h)(b)\downarrow \text{ and } U(h)(b) \in \forall_u(A,\varphi),$$

where $U(h)(b) \in \forall_u(A,\varphi)$ means that

$$\forall i \in I.\ \forall t \in (u(i) =_J j).\ U(e)(U(h)(b),t)\downarrow \text{ and } U(e)(U(h)(b),t) \in \varphi(i).$$

It is helpful to consider the underlying diagram of realizers in $\mathbb{C}$:

$$\begin{tikzcd} {[1 \rightharpoonup A] \times 1} \arrow[r, "e"] & A \\ B \times 1 \arrow[u, "h \times id"] \arrow[ur, "e \circ h \times id"'] & \end{tikzcd}$$

Let us write $\langle id, ! \rangle \colon B \to B \times 1$ for the composite $(id \times !) \circ \Delta_B$ in $\mathbb{C}$, where $! \colon B \to 1$ is the unique (total) morphism in $\mathbb{C}_t$ from B to 1. We then claim that

$$e \circ (h \times id) \circ \langle id, f \rangle \colon B \to A$$

realizes $u^*(B,\psi) = (B, \psi \circ u) \vdash (A,\varphi)$. To show the claim, let $i \in I$ and $b \in \psi(u(i))$ be arbitrary. Using the fact that $U(!)$ is total and that U preserves $\times$, we see that

$$\begin{aligned} U(e \circ (h \times id) \circ \langle id, ! \rangle)(b) &= U(e \circ (h \times id))(b, U(!)b) \\ &= U(e)(U(h)(b), U(!)b), \end{aligned}$$

which is defined and in $\varphi(i)$ by the assumptions. Hence we have shown the claim. For the other direction, suppose $h\colon B \to A$ in $\mathbb{C}$ realizes $(B, \psi \circ u) \vdash (A, \varphi)$ over I. That means that

$$\forall i \in I.\ \forall b \in \psi(u(i)).\ U(h)(b)\downarrow \text{ and } U(h)(b) \in \varphi(i).$$

Consider the following diagram in $\mathbb{C}$.

$$\begin{tikzcd}
{[1 \rightharpoonup A] \times 1} \arrow[r, "e"] & A \\
B \times 1 \arrow[u, "h' \times id"] \arrow[ur, "h \circ p_{B,1}"'] &
\end{tikzcd}$$

Here $([1 \rightharpoonup A], e)$ is the weak partial exponential of 1 and A, and thus there exists a (total) morphism $h'\colon B \to [1 \rightharpoonup A]$ in $\mathbb{C}_t$ such that the shown diagram commutes. We claim that h' realizes $(B, \psi) \vdash \forall_u(A, \varphi)$ over J. To show the claim, let $j \in J$ and $b \in \psi(j)$ be arbitrary. Note that $U(h')(b)$ is defined since h' is total. Let $i \in I$ be arbitrary and suppose $t \in u(i) =_J j$ (if there is no such t then we are trivially done). Using that U preserves $\times$ and projections, we have that

$$\begin{aligned}
U(e)(U(h')(b,t) &= U(e \circ (h' \times id))(b,t) \\
&= U(h \circ p_{B,1})(b,t) \\
&= U(h)(b),
\end{aligned}$$

which is defined and in $\varphi(i)$ by the assumption that $b \in \psi(j), t \in (u(i) =_J j)$ (so that $j = u(i)$), and h is a realizer. This completes the proof of the adjointness. The Beck-Chevalley condition holds because it holds for $\exists$, see Lemma 2.2.16.

Summarizing we have proved the following theorem.

Theorem 3.4.3. *Let $\mathbb{C}$ be a WCPC-category. Then* $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow \\ \mathbf{Set}\end{array}$ *is a split pretripos.*

For a WCPC-category $\mathbb{C}$ and a morphism $U\colon \mathbb{C} \to \mathbf{Ptl}$ of WCPC-categories, we refer to the induced pretripos $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow \\ \mathbf{Set}\end{array}$ as the **realizability pretripos over $\mathbb{C}$.**

For a realizability pretripos we have an interpretation of equality by Remark 3.4.2(ii). Working out the general approach to interpreting equality (see Chapter 2 or [Jac99]) we get the following. For two functions (terms) $u, v\colon I \to J$ in **Set**, the interpretation of the predicate

$$i\colon I \mid \emptyset \vdash u(i) =_J v(i) \tag{3.3}$$

is

$$\left(1, i \mapsto \begin{cases} U1 & \text{if } u(i) = v(i) \\ \emptyset & \text{otherwise} \end{cases}\right).$$

Hence the predicate in (3.3) is internally valid iff, for all $i \in I$, $u(i) = v(i)$ as elements of J. In other words, u and v are internally equal iff they are externally equal (*i.e.*, equal as functions in **Set**), so the equality in $\begin{matrix}\mathbf{UFam}(\mathbb{C})\\ \downarrow \\ \mathbf{Set}\end{matrix}$ is very strong.

We now present our two main example of realizability pretriposes.

Example 3.4.4. For the category **ALat** of algebraic lattices, we define $U\colon \mathbf{ALat} \to \mathbf{Ptl}$ to be the composition of the forgetful functor from **ALat** to **Set** and the inclusion functor from **Set** into **Ptl**. When we refer to the realizability pretripos $\begin{matrix}\mathbf{UFam}(\mathbf{ALat})\\ \downarrow \\ \mathbf{Set}\end{matrix}$ over **ALat** it is always understood that we refer to this functor $U\colon \mathbf{ALat} \to \mathbf{Ptl}$ just defined.

Example 3.4.5. By Proposition 3.1.18 we know that a PCA A generates a WCPC-category. When we refer to the realizability pretripos $\begin{matrix}\mathbf{UFam}(\mathbb{C}(A))\\ \downarrow \\ \mathbf{Set}\end{matrix}$ over the WCPC-category induced by A, the functor $U\colon \mathbb{C}(A) \to \mathbf{Ptl}$ is always assumed to be the functor U from Page 39. In this case, the realizability pretripos $\begin{matrix}\mathbf{UFam}(\mathbb{C}(A))\\ \downarrow p \\ \mathbf{Set}\end{matrix}$ is equivalent as a preorder fibration over **Set** to the standard realizability tripos $\begin{matrix}\mathbf{UFam}(PA)\\ \downarrow \\ \mathbf{Set}\end{matrix}$ over the partial combinatory algebra A (as defined in [HJP80], see also Chapter 5). In other words, there are fibred functors $F\colon \mathbf{UFam}(\mathbb{C}(A)) \to \mathbf{UFam}(PA)$ and $G\colon \mathbf{UFam}(PA) \to \mathbf{UFam}(\mathbb{C}(A))$ over **Set** such that $FG \cong id$ and $GF \cong id$, both vertically. Over 1, the functor F is defined by

$$F(X, \varphi \in P(UX)) = \varphi$$

(recall that $UX = \{\, X(a) \mid a \in A \,\} \subseteq A$, so φ can indeed be viewed as a subset of A and the definition thus makes sense). Over 1, the functor G is defined by

$$G(\psi) = (id_A, \psi),$$

where $id_A = \lambda x.x$. The isomorphism $G(F(X,\varphi)) \cong (X,\varphi)$ is realized by X, which is both a morphism $X \to id_A$ and a morphism $id_A \to X$ in $\mathbb{C}(A)$. The isomorphisms $F(G(\psi)) \cong \psi$ is realized by id_A (in both directions). We give a more general treatment of this example below when we consider realizability pretriposes and universal objects.

Realizability Pretriposes with Disjunction

Definition 3.4.6. Let $\mathbb{C}$ be a p-category. Then we say that $\mathbb{C}$ has a **weak initial object** if it has a weak initial object in the traditional sense, that is, if there exists an object 0 in $\mathbb{C}$ such that for all objects $X \in \mathbb{C}$, there exists a morphism $f\colon 0 \to X$ in $\mathbb{C}$.

Definition 3.4.7. Let $\mathbb{C}$ be a p-category. Then we say that $\mathbb{C}$ has **weak binary coproducts** if, for any pair of objects X and Y in $\mathbb{C}$, there exists an object $X+Y$ in $\mathbb{C}$ and a diagram $X \xrightarrow{\kappa} X+Y \xleftarrow{\kappa'} Y$ *in* $\mathbb{C}_t$ such that, for all diagrams $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\mathbb{C}$, there exists a morphism $u\colon X+Y \to Z$ in $\mathbb{C}$ such that

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[dr, "f"'] & X+Y \arrow[d, "u"] & Y \arrow[l, "\kappa'"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commutes in $\mathbb{C}$. We refer to the morphisms κ and κ' as **injections**. Note that they are required to be total.

We say that a p-category has **weak finite coproducts** if it has a weak initial object and weak binary coproducts.

In case a WCPC-category $\mathbb{C}$ has weak finite coproducts, then the weak realizability tripos over $\mathbb{C}$ has disjunction:

Theorem 3.4.8. *Let $\mathbb{C}$ be a WCPC-category with weak finite coproducts. Then* $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ *is a split pretripos with disjunction.*

Proof. By Theorem 3.4.3 we just have to show that $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ has fibred finite coproducts.

In the fibre over I, the initial object is

$$\bot = (0, \emptyset),$$

where 0 is the weak initial object.

For predicates (A, φ_1) and (B, φ_2) over I, let $A \xrightarrow{\kappa} A + B \xleftarrow{\kappa'} B$ be the weak coproduct of A and B. Then $(A, \varphi_1) \vee (B, \varphi_2)$ is defined to be

$$\big(A + B, i \mapsto \{\, U(\kappa)(a) \mid a \in \varphi_1(i)\,\} \cup \{\, U(\kappa')(b) \mid b \in \varphi_2(i)\,\}\big).$$

It is easy to verify that these definitions give split fibred finite coproducts (the totality of κ and κ' is used to show $(A, \varphi_1) \vdash (A, \varphi_1) \vee (B, \varphi_2)$ and $(B, \varphi_2) \vdash (A, \varphi_1) \vee (B, \varphi_2)$); note in particular that they are preserved under reindexing since reindexing is just composition. □

Both of our main examples, the WCPC-category $\mathbb{C}(A)$ induced by a PCA A and the category of algebraic lattices **ALat** have weak finite coproducts, as expressed by the following two propositions.

Proposition 3.4.9. *Let $(A, \cdot)$ be a partial combinatory algebra and let $\mathbb{C}(A)$ be the WCPC-category induced by $(A, \cdot)$. Then $\mathbb{C}(A)$ has weak finite coproducts.*

Proof. The weak initial object 0 is the terminal object 1 in $\mathbb{C}_t$, *i.e.*, $\lambda x.\ \mathbf{K}$. For any other object X in $\mathbb{C}$, the always undefined partial function is a morphism from 0 to X.

The weak binary coproduct of objects X and Y is $X \xrightarrow{\kappa} X + Y \xleftarrow{\kappa'} Y$ with

$$X + Y = \lambda z.\ \text{if } \pi z = \mathbf{K} \text{ then } \langle \mathbf{K}, X(\pi' z)\rangle \text{ else } \langle \mathbf{KI}, Y(\pi' z)\rangle,$$

which is A-definable in a standard way, and with

$$\kappa = \lambda x.\ \langle \mathbf{K}, Xx\rangle$$
$$\kappa' = \lambda y.\ \langle \mathbf{KI}, Yy\rangle.$$

Note that κ and κ' are both total, as required.

Suppose now given a diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\mathbb{C}$. Then

$$u = \lambda a.\ \text{if } \pi a = \mathbf{K} \text{ then } f(\pi' a) \text{ else } g(\pi' a)$$

is an A-definable partial function, which is a morphism in $\mathbb{C}$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[dr, "f"'] & X+Y \arrow[d, "u"] & Y \arrow[l, "\kappa'"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commute, as required. $\square$

Corollary 3.4.10. *Let $(A, \cdot)$ be a partial combinatory algebra and let $\mathbb{C}(A)$ be the WCPC-category induced by $(A, \cdot)$. The realizability pretripos over $\mathbb{C}(A)$ has split disjunction.*

Proposition 3.4.11. *The category* **ALat** *has weak finite coproducts.*

Remark 3.4.12. The category **ALat** does *not* have true finite coproducts because it is cartesian closed and it has the fixed point property (*i.e.*, every endomorphism has a fixed point), see [HP90]. Thus, to get the desired Corollary (3.4.14), it really is important that Proposition 3.4.8 only requires *weak* coproducts.

Proof. Any object A in **ALat** (necessarily with a non-empty underlying set) is a weak initial object: for any object B, the constant $\perp$ function from A to B is a morphism in **ALat**.

For the weak coproduct of A and B in **ALat**, view A and B as objects of $\mathbf{Top}_0$ and let $A \xrightarrow{\kappa} A + B \xleftarrow{\kappa'} B$ be the coproduct of A and B in $\mathbf{Top}_0$. Then use the embedding theorem (see, *e.g.*, [GHK+80, Lemma 3.4, (ii)]) to embed $A + B$ into an algebraic lattice C, via an embedding i. We now show that $A \xrightarrow{i\circ\kappa} C \xleftarrow{i\circ\kappa'} B$ is a weak coproduct in **ALat**. Let X be any other algebraic lattice and suppose $f\colon A \to X$ and $g\colon B \to X$ in **ALat**. View X, f and g in $\mathbf{Top}_0$. Then there exists a unique continuous function $u\colon A + B \to X$ in $\mathbf{Top}_0$ such that $f = u \circ \kappa$ and $g = u \circ \kappa'$. Now consider the following diagram in $\mathbf{Top}_0$.

$$\begin{tikzcd} A \arrow[r, "\kappa"] \arrow[ddr, "f"'] & A+B \arrow[d, hook, "i"] \arrow[dd, bend right, "u"'] & B \arrow[l, "\kappa'"'] \arrow[ddl, "g"] \\ & C \arrow[d, "v"] & \\ & X. & \end{tikzcd}$$

Since X is an algebraic lattice, it is a continuous lattice. Hence X is an injective object in $\mathbf{Top}_0$ with respect to subspace embeddings [GHK+80, Section II.3]. Thus the map $A + B \xrightarrow{u} X$ extends along the subspace inclusion $A + B \hookrightarrow C$ to a map v from C to X such that the diagram above commutes. This completes the proof of the proposition. □

Remark 3.4.13. A concrete representation of a weak coproduct

$$A \xrightarrow{i} C \xleftarrow{i'} B$$

in **ALat** of A and B can be obtained by letting $C = Sigma^2 \times A \times B$ $(\Sigma = \{\bot \leq \top\})$, and letting

$$\begin{aligned} i\colon A \to C &= x \mapsto ((\bot, \top), x, \bot_B) \\ i'\colon B \to C &= x \mapsto ((\top, \bot), \bot_A, x). \end{aligned}$$

Corollary 3.4.14. *The realizability pretripos over* **ALat** *has split disjunction.*

Proof. By Theorem 3.4.8 and Proposition 3.4.11. □

Realizability Pretriposes and Universal Objects

Example 3.4.5 is an instance of a more general phenomenon, which we now describe. For the remainder of this subsection we assume that the reader is familiar with tripos theory, see Chapter 5. Let us say that a category $\mathbb{C}$ has a **universal object** V if all objects X in $\mathbb{C}$ are retracts of V. Observe that $\mathbb{C}(A)$ has a universal object, namely id_A (as implicitly explained in Example 3.4.5). We can now show that (when U is faithful) the realizability pretripos over such a category $\mathbb{C}$ has a generic object just in case $\mathbb{C}$ has a universal object.

Theorem 3.4.15. *Let $\mathbb{C}$ be a WCPC-category and suppose $U\colon \mathbb{C} \to$* **Ptl** *is a faithful functor. Then $\mathbb{C}$ has a universal object if and only if* $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow \\ \mathbf{Set}\end{array}$ *has a generic object.*

Proof. Suppose $\mathbb{C}$ has a universal object V. We then claim that

$$(V, id\colon P(UV) \to P(UV))$$

is a generic object for $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$. To show the claim, suppose that $(A, \varphi\colon I \to PUA)$ is any predicate in the fibre over I. Then by the assumption on $\mathbb{C}$, the object A is a retract of V, that is, there are morphisms r and s in $\mathbb{C}$

$$A \underset{s}{\overset{r}{\rightleftarrows}} V$$

such that $rs = id_A$. Since $U(rs) = U(r)U(s) = U(id) = id$, we have that $U(s)$ is a total function. Thus $P(Us)$ is a morphism from $P(UA)$ to $P(UV)$ in **Set**. We now show that $(A, \varphi) \cong (P(Us) \circ \varphi)^*(V, id)$ in the fibre over I, which completes the proof of the claim. Recall that $(P(Us) \circ \varphi)^*(V, id) = (V, P(Us) \circ \varphi)$. Clearly s is a realizer for $(A, \varphi) \vdash (V, P(Us) \circ \varphi)$, since $U(s)$ is total. For the other direction, note that r is a realizer for $(V, P(Us) \circ \varphi) \vdash (A, \varphi)$, since for all $i \in I$, all $a \in P(Us)(\varphi(i))$, we have that $a = U(s)(b)$ for some $b \in \varphi$, so $U(r)(a) = U(r)(U(s)(b)) = b \in \varphi$.

For the other direction suppose that $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ is a that there is a weak generic object $(W, \psi\colon V \to P(UW))$, over some $V \in \mathbf{Set}$. Let $A \in \mathbb{C}$ be arbitrary. We show that A is a retract of W to conclude that W is a universal object in $\mathbb{C}$. Consider the predicate (A, φ) over $U(A)$, with $\varphi\colon U(A) \to P(UA)$ the function $a \mapsto \{a\}$. Since (W, ψ) is a weak generic object, there exists a morphism $f\colon U(A) \to V$ in **Set**, such that

$$(A, \varphi) \cong f^*(W, \psi)$$

in the fibre over UA. Hence there are morphisms $s\colon A \to W$ and $r\colon W \to A$ in $\mathbb{C}$ such that

$$\forall a \in UA.\ \forall b \in \varphi(a).\ U(s)(b)\downarrow \text{ and } U(s)(b) \in \psi(f(a))$$

and

$$\forall a \in UA.\ \forall b \in \psi(f(a)).\ U(r)(b)\downarrow \text{ and } U(r)(b) \in \varphi(a).$$

Recalling that $\varphi(a) = \{a\}$ we get that

$$\forall a \in UA.\ U(r)(U(s)(a))\downarrow \text{ and } U(r)(U(s)(a)) \in \varphi(a) = \{a\}$$

so

$$\forall a \in UA.\ U(rs)(a) = a$$

and thus $U(rs) = id_{UA}$. Since U is faithful by assumption, we conclude that $rs = id_A$ in $\mathbb{C}$, thus completing the proof that A is a retract of W. $\square$

Corollary 3.4.16. *Let $\mathbb{C}$ be a WCPC-category with weak finite coproducts and suppose $U : \mathbb{C} \to \mathbf{Ptl}$ is a faithful functor. Then $\mathbb{C}$ has a universal object if and only if* $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ *is a tripos.*

Proof. Immediate from Theorems 3.4.15 and 3.4.8. □

Remark 3.4.17. For $\mathbb{C}$ be a WCPC-category, the tripos-to-topos construction applied to $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ yields a category which is equivalent to the exact completion $(\mathbf{Asm}(\mathbb{C}))_{\mathrm{ex/reg}}$ of $\mathbf{Asm}(\mathbb{C})$. Indeed $\mathbf{Asm}(\mathbb{C})$ is a reflective subcategory of $(\mathbf{Asm}(\mathbb{C}))_{\mathrm{ex/reg}}$. Hence some of the results we present in the following (such as Theorem 3.5.1) could also be obtained by showing that $(\mathbf{Asm}(\mathbb{C}))_{\mathrm{ex/reg}}$ has certain good properties which are then reflected down to $\mathbf{Asm}(\mathbb{C})$. For concreteness, however, we have decided to show directly that $\mathbf{Asm}(\mathbb{C})$ has the properties we are interested in. For more remarks on the relation to categories obtained via exact completions, see Section 3.8.

In fact, we can strengthen the above results to characterize exactly when the tripos-to-topos construction applied to $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ yields a topos. To do so we apply a result of Pitts who characterized (in 1982) when the tripos-to-topos construction applied to a pretripos with disjunction yields a topos [Pit99]. (Pitts calls a pretripos with disjunction a hyperdoctrine.) Pitts showed in particular that the tripos-to-topos construction applied to the pretripos $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ (for $\mathbb{C}$ a WCPC-category) yields a topos iff the following axiom holds:[5]

Axiom 3.4.18 (CA). *For all sets X there is a set PX and a predicate $In_X \in \mathbf{UFam}(\mathbb{C})_{X\times PX}$ such that, for any set I and $R \in \mathbf{UFam}(\mathbb{C})_{X\times I}$, the sentence*

$$\forall i \colon I.\ \exists s \colon PX.\ \forall x \colon X.\ \mathrm{In}_X(x,s) \supset\subset R(x,i)$$

is satisfied in $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$.

[5] In fact, Pitts assumed that the pretripos has disjuction, but going through his proof one sees that disjunctions are not needed.

Theorem 3.4.19. *Let $\mathbb{C}$ be a WCPC-category and suppose $U\colon \mathbb{C} \to$ **Ptl** is a faithful functor. Then $\mathbb{C}$ has a universal object if and only if* $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ *satisfies the axiom (CA).*

Proof. By Theorem 3.4.15, if $\mathbb{C}$ has a universal object, then $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{array}$ is a tripos and thus it also satisfies the axiom (CA).

For the other direction, suppose that $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{array}$ satisfies the axiom (CA). Let $X = 1$ and let Σ denote the object PX in (CA). Let $V \in \mathbb{C}$ be the underlying object of $\mathrm{In} = \mathrm{In}_1$, which exists *qua* (CA). We show that V is a universal object in $\mathbb{C}$. Let $A \in \mathbb{C}$ be arbitrary, let $I = UA$ and let $R(a) = \{a\}$. By (CA) the following sentence is valid in the realizability pretripos:

$$\forall a\colon UA.\ \exists s\colon \Sigma.\ \mathrm{In}(s) \supset\!\subset R(a).$$

Thus there is a realizer in

$$\bigcap_{a\in UA}\left(\bigcup_{s\in\Sigma}\mathrm{In}(s) \supset\!\subset R(a)\right) \subseteq U([V \rightharpoonup A]\times[A \rightharpoonup V]).$$

It follows, as in the proof of Theorem 3.4.15, that A is a retract of V. $\square$

Remark 3.4.20. When I first tried to show the theorem above (using the same idea as is currently employed in the proof), I failed to see that what I had written down was actually a proof. It was not until I saw a very similar proof of Peter Lietz and Thomas Streicher [Lie] that I realized that it worked. See Section 3.9 for a description of the relation of our work to the work of Lietz and Streicher.

Corollary 3.4.21. *For $\mathbb{C}$ be a WCPC-category and $U\colon \mathbb{C} \to$ **Ptl** a faithful functor, the tripos-to-topos construction applied to* $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ *yields a topos iff $\mathbb{C}$ has a universal object.*

A further corollary is that the tripos-to-topos construction applied to the realizability tripos induced by the category of algebraic lattices does not produce a topos, since clearly **ALat** does not have a universal object (for cardinality reasons).

Realizability Preriposes and Splitting of Total Idempotents

One may also observe that $\begin{array}{c}\mathbf{UFam}(\mathbb{C}(A))\\ \downarrow\\ \mathbf{Set}\end{array}$ is equivalent, as a preorder fibration over **Set**, to $\begin{array}{c}\mathbf{UFam}(\mathrm{M}(A))\\ \downarrow\\ \mathbf{Set}\end{array}$, where $\mathrm{M}(A)$ is the monoid of partial A-definable functions. This observation is also an instance of a more general phenomenon, which we now describe.

Let $\mathbb{C}$ be a category and let $U\colon \mathbb{C} \to \mathbf{Ptl}$ be a functor. Define $\mathrm{Split}(\mathbb{C}, U)$ to be the category with objects the idempotents of $\mathbb{C}$ that are mapped by U into **Set** and with morphisms $f\colon X \to Y$ morphisms f in $\mathbb{C}$ satisfying that $YfX = f$. The identity on X is X itself and composition of morphisms is composition in $\mathbb{C}$. The functor U induces a functor $\mathrm{Split}(U)\colon \mathrm{Split}(\mathbb{C}, U) \to$ **Ptl**[6] defined by setting $\mathrm{Split}(U)(X)$ equal to the image of $U(X)$ for an object X (recall that X is an idempotent $X\colon A \to A$ in $\mathbb{C}$) and by setting $\mathrm{Split}(U)(f) = f$ for a morphism f. Observe that $\mathbb{C}(A)$ is $\mathrm{Split}(\mathrm{M}(A), U_0)$ for $U_0\colon \mathrm{M}(A) \to \mathbf{Ptl}$ the inclusion functor and that $U\colon \mathbb{C}(A) \to \mathbf{Ptl}$ as defined on Page 39 is $\mathrm{Split}(U_0)$.

Proposition 3.4.22. *Let $\mathbb{C}$ be a category and let $U\colon \mathbb{C} \to \mathbf{Ptl}$ be a functor. Then $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Set}\end{array}$ is equivalent, as a preorder fibration, to $\begin{array}{c}\mathbf{UFam}(\mathrm{Split}(\mathbb{C}))\\ \downarrow\\ \mathbf{Set}\end{array}$.*

Proof. The proof is essentially as in Example 3.4.5 (define functors F and G in the same manner to prove the equivalence). □

3.5 Assemblies over Pretriposes

Let $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{array}$ be a pretripos. Recall from Remark 3.4.2(ii) that p then in particular is a regular fibration. Therefore we may construct the category of assemblies, $\mathbf{Asm}(p)$, over it. In this section we show that $\mathbf{Asm}(p)$ is locally cartesian closed. Moreover, we show that if p has disjunction, then $\mathbf{Asm}(p)$ has finite coproducts.

Theorem 3.5.1. *Let $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{array}$ be a pretripos. Then $\mathbf{Asm}(p)$ is locally cartesian closed.*

[6] The functor $\mathrm{Split}(U)$ really arises because $\mathrm{Split}(-)$ in a suitable sense is a 2-functor and $\mathrm{Split}(\mathbf{Ptl}, id)$ is equivalent to **Ptl**.

Proof. By Proposition 3.3.3 we already know that $\mathbf{Asm}(p)$ is regular, so it suffices to consider local closure. It suffices to show, see, *e.g.*, [Jac99, Proposition 1.9.8], that for each morphism $u\colon (I, E_I) \to (J, E_J)$, the pullback functor

$$u^*\colon \mathbf{Asm}(p)/(J, E_J) \to \mathbf{Asm}(p)/(I, E_I)$$

has a right adjoint $\prod_u$. We define $\prod_u$ as follows. Let $\varphi\colon (X, E_X) \to (I, E_I)$ be an object over (I, E_I). Then set $\prod_u(\varphi) \stackrel{\text{def}}{=} \psi$, where the domain of ψ is

$$\Big(\coprod_{j\in J} \Big\{ f\colon \{ i \in I \mid u(i) = j \} \to X \mid \varphi(f(i)) = i \text{ and } E'_j(f) \text{ valid in } p \Big\}, E \Big),$$

and

$$\begin{aligned} E'_j(f) &\stackrel{\text{def}}{=} \forall i\colon \{ i \in I \mid u(i) = j \}.\ E_I(i) \supset E_X(f(i)) \\ E(j, f) &\stackrel{\text{def}}{=} E_J(j) \wedge E'_j(f) \\ \psi(j, f) &\stackrel{\text{def}}{=} j. \end{aligned}$$

It is easy to see that the domain of ψ is a well-defined object and that ψ is a well-defined morphism into (J, E_J).

For a morphism $f\colon \varphi \to \varphi'$ in $\mathbf{Asm}(p)/(I, E_I)$, we define

$$\textstyle\prod_u(f) \stackrel{\text{def}}{=} (j, g) \mapsto (j, f \circ g).$$

It is straightforward to verify that this operation is well-defined.

Let φ and $\psi = \prod_u(\varphi)$ be as above and suppose $\zeta\colon (Y, E_Y) \to (J, E_J)$. We are to show (confusing an object in the slice category with its domain) that

$$\begin{array}{cl} u^*(Y, E_Y) \longrightarrow (X, E_X) & \text{in } \mathbf{Asm}(p)/(I, E_I) \\ \hline\hline (Y, E_Y) \longrightarrow \prod_u(X, E_X) & \text{in } \mathbf{Asm}(p)/(J, E_J), \end{array}$$

where

$$u^*(Y, E_Y) = (\{ (i, y) \mid u(i) = \zeta(y) \}, E_I \wedge E_Y).$$

Suppose $f\colon u^*(Y, E_Y) \to (X, E_X)$. Then define its transpose to be $\hat{f} = y \mapsto (\zeta(y), \lambda i.\ f(i, y))$. Note that $\hat{f}$ is a morphism: under the assumption that $E_Y(y)$, we are to show that $E(\hat{f}(y))$, that is, $E(\zeta(y), \lambda i.\ f(i, y)) = E_J(\zeta(y)) \wedge E'_{\zeta(y)}(f)$. But $E_J(\zeta(y))$ holds as ζ is a morphism and

$$y\colon Y \mid E_Y(y) \vdash E'_{\zeta(y)}(f) = \forall i\colon \{ i \in I \mid u(i) = \zeta(y) \}.\ E_I(i) \supset E_X(f(i, y))$$

holds by the assumption on f. Thus $\hat{f}$ is a morphism in $\mathbf{Asm}(p)$. Moreover, it is a morphism *over* (J, E_J) because clearly $\psi\hat{f} = \zeta$.

For the other direction, suppose that $g\colon (Y, E_Y) \to \prod_u(X, E_X)$ over (J, E_J). Then define $\breve{g}\colon u^*(Y, E_Y) \to (X, E_X)$ by $(i, y) \mapsto \pi'(g(y))(i)$. Note that the application of $\pi'(g(y))$ to i is well-defined, that is, i is really in the domain of $\pi'(g(y))$ because $u(i) = \zeta(y)$ and $\pi(g(y)) = \psi(g(y)) = \zeta(y)$ since g is a map from ζ to ψ over (J, E_J). To verify that $\breve{g}$ is a well-defined morphism in $\mathbf{Asm}(p)$ we are to show that

$$\{\, (i, y) \in I \times Y \mid u(i) = \zeta(y) \,\} \mid E_I(i) \wedge E_Y(y) \vdash E_X(\pi'(g(y))(i)),$$

but this holds because g is a morphism. Finally, $\breve{g}$ is a morphism over (I, E_I), that is, $\varphi\circ\breve{g} = u^*(\zeta)$ because $u^*(\zeta)(i, y) = i$ and $\varphi\circ\breve{g}(i, y) = \varphi(\pi'(g(y))(i)) = i$ since $\pi'(g(y))(i) \in \varphi^{-1}(i)$ by assumption.

We have now defined the transposes. It is straightforward to see that they constitute an isomorphism, that is, $\breve{\hat{f}} = f$ and $\hat{\breve{g}} = g$ (for the latter use that g is a map in the slice category). The correspondence is natural and we thus have the required adjunction. $\square$

Remark 3.5.2. For $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{array}$ a pretripos, the functor $\nabla\colon \mathbf{Set} \to \mathbf{Asm}(p)$ (see Definition 3.3.4) preserves exponentials and $\mathbf{Set}$ is an exponential ideal of $\mathbf{Asm}(p)$ since $\mathbf{Set}$ is a full reflective subcategory and the reflector U preserves finite products, see Propositions 3.3.5 and 3.3.6.

Theorem 3.5.3. *Let* $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbf{Set}\end{array}$ *be a pretripos with disjunction. Then* $\mathbf{Asm}(p)$ *has finite coproducts.*

Proof. The initial object is $(\emptyset, \bot_\emptyset)$ (trivially an object in $\mathbf{Asm}(p)$ since there are no global elements). For any object (X, E_X), the unique function $!$ from $\emptyset$ to X in $\mathbf{Set}$ is also (the unique) morphism in $\mathbf{Asm}(p)$ from $(\emptyset, \bot_\emptyset)$ to (X, E_X), since trivially, $x\colon \emptyset \mid \bot_\emptyset \vdash E_X(!(x))$.

The coproduct of (X, E_X) and (Y, E_Y) is $(X + Y, E)$, where

$$X \xrightarrow{\kappa} X + Y \xleftarrow{\kappa'} Y$$

is the usual set-theoretic coproduct and where

$$\begin{aligned} v\colon X + Y \mid E(v) \stackrel{\text{def}}{=} \; & \big(\exists x\colon X.\ v = \kappa(x) \wedge E_X(x)\big) \vee \\ & \big(\exists y\colon Y.\ v = \kappa'(y) \wedge E_Y(y)\big). \end{aligned}$$

Note that $(X+Y,E)$ is indeed a well-defined object and that clearly

$$\kappa\colon (X,E_X) \to (X+Y,E) \qquad \text{and} \qquad \kappa\colon (Y,E_Y) \to (X+Y,E)$$

are morphisms in $\mathbf{Asm}(p)$. Suppose

$$f\colon (X,E_X) \to (Z,E_Z) \quad \text{and} \quad g\colon (Y,E_Y) \to (Z,E_Z).$$

Then there exists a unique map $u\colon X+Y \to Z$ in **Set** such that $u \circ \kappa = f$ and $u \circ \kappa' = g$. It suffices to verify that u is also a map $(X+Y,E) \to (Z,E_Z)$ in $\mathbf{Asm}(p)$, *i.e.*, that

$$v\colon X+Y \mid E(v) \vdash E_Z(u(v))$$

holds in the logic of p. But that is easy to see, arguing in the internal language of p and using that external equality implies internal equality. $\square$

It follows from Corollary 3.4.10 that the category of assemblies constructed over the realizability pretripos over the WCPC-category induced by a PCA has finite coproducts. Likewise, by Corollary 3.4.14 we get that the category of assemblies constructed over the realizability pretripos over the category of algebraic lattices has finite coproducts.

3.6 Assemblies over Realizability Pretriposes

Let $\mathbb{C}$ be a WCPC-category and let $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow p\\ \mathbf{Set}\end{array}$ be the realizability pretripos over $\mathbb{C}$, *cf.* Section 3.4.1. In the preceding section we saw that $\mathbf{Asm}(p)$ then is locally cartesian closed. Referring to Seely's [See84] seminal work, we therefore have a model of dependent type theory. However, there are some problems with interpreting dependent type theory directly in locally cartesian closed categories. The main problem is to make sure that the actual interpretation function is well-defined, which is problematic because the substitution functors (pullback functors) do not commute on the nose but only up to (canonical) isomorphism. See for example, [Luo94, Pit95, Mog95, Reu95, Hof94] for a more thorough discussion of this issue.

We shall therefore make a point of describing a model of dependent type theory in a so-called "split" way, so as to avoid the problems with interpreting dependent type theory.

For the technical development, we make use of B. Jacobs' fibrational description of models of dependent type theory [Jac91, Jac93, Jac99], which is related to display-map categories [Tay86, HP89], categories with attributes [Car78, Mog91], D-categories [Ehr88], and thorough fibrations [Pav90]. See [Jac93] or [Jac99] for a comprehensive introduction. We recall the needed definitions below in Section 3.6.1 and then in Section 3.6.2 we show how to define a *split* fibration which is equivalent to the codomain on $\mathbf{Asm}(p)$ and which thus models dependent type theory. Our treatment generalizes the results for assemblies and modest sets over partial combinatory algebras (see, for example, [Jac99]) and assemblies and modest sets over algebraic lattices [BBS98], see Examples 3.6.10 and 3.6.11 in Section 3.6.2.

In Appendix A we present the calculus of dependent type theory and sketch how it is interpreted in very concrete terms for the particular case of modest sets over algebraic lattices (see Section 3.7 for modest sets). For the reader who is not familiar with closed comprehension categories and the interpretation of dependent type theory in such, it may be useful to read Appendix A in parallel with the more abstract treatment in this section.

In Subsection 3.6.3 we show that for assemblies over *realizability* pretriposes with disjunction, the finite coproducts in $\mathbf{Asm}(p)$ (see Theorem 3.5.3 in the previous section) are disjoint and stable under pullback.

3.6.1 Comprehension Categories

In this subsection we recall the notion of a comprehension category and some accompanying definitions from [Jac99, Section 9.3] and [Jac93]. In the mentioned references, the connection between comprehension categories and so-called weakening and contraction comonads is described in detail; we shall not need those concepts here. We remark that the definition of a comprehension category is inspired by Lawvere's categorical notion of comprehension [Law68].

Definition 3.6.1. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration and let $\mathcal{P}\colon \mathbb{E} \to \mathbb{B}^{\to}$ be a functor satisfying

1. $\mathrm{cod} \circ \mathcal{P} = p\colon \mathbb{E} \to \mathbb{B}$;
2. for each cartesian map f in $\mathbb{E}$, the induced square $\mathcal{P}(f)$ in $\mathbb{B}$ is a pullback.

Such a functor $\mathcal{P}$ will be called a **comprehension category** (on p). We shall often write $\{-\} = \mathrm{dom} \circ \mathcal{P}\colon \mathbb{E} \to \mathbb{B}$. Thus $\mathcal{P}$ is a natural transformation $\{-\} \Rightarrow p$.

Such a comprehension category $\mathcal{P}$ will be called **full** if $\mathcal{P}$ is a full and faithful functor $\mathbb{E} \to \mathbb{B}^{\to}$. And it is called **split** (or **cloven**), whenever the fibration $\mathrm{cod} \circ \mathcal{P} = p$ is split (or cloven).

Note that it is not required that the base category $\mathbb{B}$ has *all* pullbacks.

Definition 3.6.2. Consider a comprehension category $\mathcal{P}\colon \mathbb{E} \to \mathbb{B}^{\to}$ on $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ and a fibration $\begin{array}{c}\mathbb{D}\\ \downarrow q\\ \mathbb{E}\end{array}$ in a situation

$$\begin{tikzcd} & \mathbb{D} \arrow[d, "q"] \\ \mathbb{E} \arrow[r, bend left, "\{-\}"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow\mathcal{P}"] & \mathbb{B}. \end{tikzcd}$$

1. We say that q has $\mathcal{P}$-products (resp. coproducts) if there is for each $X \in \mathbb{E}$ an adjunction

   $$(\mathcal{P}X)^* \dashv \textstyle\prod_X \qquad (\text{resp. } \coprod_X \dashv (\mathcal{P}X)^*)$$

   plus a Beck-Chevalley condition: for each cartesian map $f\colon X \to Y$ in $\mathbb{E}$, the canonical natural transformation

   $$(pf)^* \textstyle\prod_Y \Rightarrow \prod_X \{f\}^* \qquad (\text{resp. } \coprod_X \{f\}^* \Rightarrow (pf)^* \coprod_Y)$$

   is an isomorphism.

2. We say that q has $\mathcal{P}$-equality if for each $X \in \mathbb{E}$ there is an adjunction

   $$\mathrm{Eq}_X \dashv \delta_X^*,$$

   where δ_X is the unique mediating diagonal

   $$\langle id, id\rangle\colon \{X\} \to \{(\mathcal{P}X)^*(X)\}$$

   in

   $$\begin{tikzcd} \{X\} \arrow[dr, dashed, "\delta_X"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\ & \{(\mathcal{P}X)^*(X)\} \arrow[r, "\pi'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \{X\} \arrow[d, "\mathcal{P}X"] \\ & \{X\} \arrow[r, "\mathcal{P}X"'] & pX \end{tikzcd}$$

where $\pi = \mathcal{P}((\mathcal{P}\,X)^*(X))$ and $\pi' = \{\overline{\mathcal{P}\,X}(X)\}$ are the pullback projections. Additionally, there is a Beck-Chevalley requirement: for each cartesian map $f\colon X \to Y \in \mathbb{E}$, the canonical natural transformation

$$\mathrm{Eq}_X\{f\}^* \Rightarrow \{f'\}^*\mathrm{Eq}_Y$$

should be an isomorphism—where f' is the unique morphism in $\mathbb{E}$ over $\{f\}$ in

$$\begin{tikzcd}
(\mathcal{P}\,X)^*(X) \arrow[r, dashed, "f'"] \arrow[d] & (\mathcal{P}\,Y)^*(Y) \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Example 3.6.3. Let $\mathbb{B}$ be a category with finite limits. Then $id\colon \mathbb{B}^{\to} \to \mathbb{B}^{\to}$ is a comprehension category on the codomain fibration $\begin{smallmatrix}\mathbb{B}^{\to}\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$. Products and coproducts with respect to this comprehension category are products and coproducts along all morphisms in $\mathbb{B}$. The diagonal on a family $X \to I$ is the mediating map $X \to X \times_I X$.

Convention 3.6.4. Let $\mathcal{P}\colon \mathbb{E} \to \mathbb{B}^{\to}$ be a comprehension category and write $p = \mathrm{cod} \circ \mathcal{P}\colon \mathbb{E} \to \mathbb{B}$ for the fibration involved. For an object $X \in \mathbb{E}$, the corresponding morphism $\mathcal{P}\,X$ in $\mathbb{B}$ will be called a **projection** or a **display map**. We therefore often write π_X for $\mathcal{P}\,X$, when the functor $\mathcal{P}$ is understood from context. An induced reindexing functor $\pi_X^* = (\mathcal{P}\,X)^*$ will be called a **weakening functor**.

Definition 3.6.5. A fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ with a terminal object functor $1\colon \mathbb{B} \to \mathbb{E}$ is said to admit **comprehension** if this functor 1 has a right adjoint, which we commonly write $\{-\}\colon \mathbb{E} \to \mathbb{B}$. We then have adjunctions

$$p \dashv 1 \dashv \{-\}.$$

In this situation we get a functor $\mathbb{E} \to \mathbb{B}^{\to}$ by $X \mapsto p(\epsilon_X)$, where ϵ_X is the counit $1\{X\} \to X$ of the adjunction $(1 \dashv \{-\})$ at X. This functor is actually a comprehension category, see [Jac99, Page 574] or [Jac93]. In such a situation, we shall call this functor a **comprehension category with unit**. And we shall say that p admits **full comprehension** if this induced comprehension category is full (*i.e.*, $\mathbb{E} \to \mathbb{B}^{\to}$ is a full and faithful functor).

Definition 3.6.6. Let $\mathcal{P}\colon \mathbb{E} \to \mathbb{B}^{\to}$ be a comprehension category. We say that $\mathcal{P}$ has **products** if its underlying fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ —where $p = \mathrm{cod} \circ \mathcal{P}$— has products with respect to the comprehension category $\mathcal{P}\colon \mathbb{E} \to \mathbb{B}^{\to}$, see Definition 3.6.1.

Similarly we say that $\mathcal{P}$ has **coproducts** if the fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$ has coproducts with respect to $\mathcal{P}$.

And $\mathcal{P}$ has equality if $\begin{smallmatrix}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$ has equality with respect to $\mathcal{P}$.

Definition 3.6.7. Let $\mathcal{P}\colon \mathbb{E} \to \mathbb{B}^{\to}$ be a comprehension category.

1. We say that $\mathcal{P}$ has **strong coproducts** if it has coproducts as above in such a way that the canonical maps κ are isomorphisms in

$$\begin{tikzcd}
\{Y\} \arrow[r, "\kappa", "\cong"'] \arrow[d, "\pi"'] & \{\coprod_X(Y)\} \arrow[d, "\pi"] \\
\{X\} \arrow[r, "\pi"'] & pX.
\end{tikzcd}$$

2. Similarly, $\mathcal{P}$ has **strong equality** if we have canonical isomorphisms

$$\begin{tikzcd}
\{Y\} \arrow[r, "\kappa", "\cong"'] \arrow[d, "\pi"'] & \{\mathrm{Eq}_X(Y)\} \arrow[d, "\pi"] \\
\{X\} \arrow[r, "\delta"'] & \{\pi_X^*(X)\}.
\end{tikzcd}$$

The canonical maps $\{Y\} \to \{\coprod_X(Y)\}$ and $\{Y\} \to \{\mathrm{Eq}_X(Y)\}$ arise by applying the functor $\{-\} = \mathrm{dom} \circ \mathcal{P}$ to the composites

$$Y \xrightarrow{\eta} \pi_X^* \textstyle\coprod_X(Y) \longrightarrow \coprod_X(Y) \qquad \text{and} \qquad Y \xrightarrow{\eta} \delta_X^* \mathrm{Eq}_X(Y) \longrightarrow \mathrm{Eq}_X(Y).$$

Definition 3.6.8. A **closed comprehension category** (CCompC) is a full comprehension category with unit, which has products and strong coproducts, and which has a terminal object in its base category. It will be called split if all of its fibred structure is split.

For details of how to interpret dependent type theory in a closed comprehension category (and a detailed account of the exact rules of dependent type theory), see [Jac99]. See also Appendix A for a concrete example.

3.6.2 A Model of Dependent Type Theory

Let $\mathbb{C}$ be a WCPC-category and let $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow p\\ \mathbf{Set}\end{array}$ be the realizability pretripos over $\mathbb{C}$. Note that an object in $\mathbf{Asm}(p)$ is of the form $(X, (A, E))$, where X is an object in $\mathbf{Set}$ and (A, E) is an object in $\mathbf{UFam}(\mathbb{C})_X$, that is, A is an object in $\mathbb{C}$ and E is a function $X \to P(UA)$.

Convention 3.6.9. We write objects $(X, (A, E)) \in \mathbf{Asm}(p)$ as triples (X, A, E), leaving out a pair of parentheses.

Note that the condition on objects (X, A, E) in $\mathbf{Asm}(p)$ regarding global elements now simplifies to the familiar [Lon94, BBS98] condition $\forall x \in X.\ E(x) \neq \emptyset$. Indeed the following two examples show how our setup specializes to the cases of assemblies over PCA's and over algebraic lattices.

Example 3.6.10. Let A be a PCA. Recall from Example 3.4.5 that the induced realizability pretripos $\begin{array}{c}\mathbf{UFam}(\mathbb{C}(A))\\ \downarrow p\\ \mathbf{Set}\end{array}$ is equivalent to the standard realizability tripos over a A. Since the first-order structure is defined categorically, it is preserved by the equivalence functors. From the preceding it is easy to verify that the category $\mathbf{Asm}(p)$ of assemblies over p is equivalent to the usual category of assemblies $\mathbf{Asm}(A)$ over A as defined, *e.g.*, in [Lon94].

Example 3.6.11. Let $\begin{array}{c}\mathbf{UFam}(\mathbf{ALat})\\ \downarrow p\\ \mathbf{Set}\end{array}$ be the realizability pretripos over $\mathbf{ALat}$. Then $\mathbf{Asm}(p)$ is equivalent to the category of assemblies over $\mathbf{ALat}$ as defined in [BBS98].

Convention 3.6.12. For $\mathbb{C}$ a WCPC-category, we often write $\mathbf{Asm}(\mathbb{C})$ for $\mathbf{Asm}(p)$ where p is the realizability pretripos over $\mathbb{C}$. (This choice of notation is in accordance with [BBS98], where we write $\mathbf{Asm}(\mathbf{ALat})$ for the category of assemblies over $\mathbf{ALat}$.)

We now proceed to show how to define a split closed comprehension category for the category of assemblies over a realizability pretripos, thus obtaining a model of dependent type theory. Our definitions and results are generalizations of corresponding results for assemblies over partial combinatory algebras and assemblies over algebraic lattices, see [Jac99] and [BBS98].

Convention 3.6.13. For the remainder of this section, let $\mathbb{C}$ be a WCPC-category and let $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow p\\ \mathbf{Set}\end{array}$ be the realizability pretripos over $\mathbb{C}$.

Definition 3.6.14. Define $\mathbf{UFam}(\mathbf{Asm}(p))$ to be the category with

objects triples $(I, A, (X_i, E_i)_{i \in X_I})$, where

$$I = (X_I, A_I, E_I) \in \mathbf{Asm}(p),$$

and $(X_i, A, E_i) \in \mathbf{Asm}(p)$, for all $i \in X_I$.

morphisms $(I, A, (X_i, E_i)_{i \in X_I}) \to (J, B, (Y_j, E'_j)_{j \in X_J})$, with

$$I = (X_I, A_I, E_I) \qquad \text{and} \qquad J = (X_J, A_J, E_J),$$

are pairs

$$(f, (f_i)_{i \in X_I})$$

such that $f : I \to J$ in $\mathbf{Asm}(p)$ and such that

$$\emptyset \mid \emptyset \vdash \forall i : X_I.\ \forall x : X_i.\ E_I(i) \supset (E_i(x) \supset E'_{f(i)}(f_i(x))) \tag{3.4}$$

is valid in the logic of p.

The identity on object $(I, A, (X_i, E_i)_{i \in X_I})$ with $I = (X_I, A_I, E_I)$, is

$$(id, (id_i)_{i \in X_I}).$$

The composition of $(f, (f_i)_{i \in X_I})$ and $(g, (g_j)_{j \in X_J})$ is $(g \circ f, (g_{f(i)} \circ f_i)_{i \in X_I})$.

Remark 3.6.15. Categorically, the second quantifier in (3.4) is given as $\forall_\pi$ with π the projection $\coprod_{i \in X_I} X_i \to X_I$ in $\mathbf{Set}$. The $\mathbf{U}$ in $\mathbf{UFam}(\mathbf{Asm}(p))$ refers to the fact that a family $(I, A, (X_i, E_i)_{i \in X_I})$ over an object I is *uniform* in the sense that all the existence predicates E_i have the *same* underlying object $A \in \mathbb{C}$ as object of realizers.

Proposition 3.6.16. *The forgetful functor* $\mathbf{UFam}(\mathbf{Asm}(p)) \to \mathbf{Asm}(p)$ *given by* $(I, A, (X_i, E_i)_{i \in X_I}) \mapsto I$ *and* $(f, (f_i)_{i \in X_I}) \mapsto f$ *is a split fibration which is equivalent, as a fibration, to the codomain fibration over* $\mathbf{Asm}(p)$.

Proof. Let $(J, B, (Y_j, E'_j)_{j \in X_J})$, with $J = (X_J, A_J, E_J)$, be an object in the category $\mathbf{UFam}(\mathbf{Asm}(p))$ and suppose $u : I \to J$ in $\mathbf{Asm}(p)$ with $I = (X_I, A_I, E_I)$. Then we can form a family $u^*(J, B, (Y_j, E'_j)_{j \in X_J})$ over I as $(I, B, (Y_{u(i)}, E'_{u(i)})_{i \in X_I})$. There is then an associated cartesian lifting

$$(u, (id)_{i \in X_I}) : (I, B, (Y_{u(i)}, E'_{u(i)})_{i \in X_I}) \to (J, B, (Y_j, E'_j)_{j \in X_J})$$

over u. This choice of liftings forms a splitting.

Define the functor $\mathcal{P}$ as in

$$\begin{tikzcd}
\mathbf{UFam}(\mathbf{Asm}(p)) \arrow[rr, "\mathcal{P}"] \arrow[dr] & & \mathbf{Asm}(p)^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathbf{Asm}(p) &
\end{tikzcd}$$

by mapping an object $(I, A, (X_i, E_i)_{i \in X_I})$, with $I = (X_I, A_I, E_I)$, to

$$(\textstyle\coprod_{i \in X_I} X_i, A_I \times A, E) \xrightarrow{\pi} I,$$

with

$$(i, x)\colon \textstyle\coprod_{i \in X_I} X_i \vdash E(i, x) \stackrel{\mathrm{def}}{=} E_I(i) \wedge E_i(x).$$

(In more detail, note that $E_i\colon X_i \to P(UA)$ so may be viewed as a function $\coprod_{i \in X_I} X_i \to P(U(A))$, $(i, x) \mapsto E_i(x)$, *i.e.*, a predicate in the fibre over $\coprod_{i \in X_I} X_i$ in the fibration p. Furthermore, $E_I\colon X_I \to P(UA_I)$ is a predicate over X_I, so may be reindexed to a predicate $\pi^*(E_I)$ over $\coprod_{i \in X_I} X_i$ (where $\pi\colon \coprod_{i \in X_I} X_i \to X_I$). Finally, E can be defined as the predicate over $\coprod_{i \in X_I} X_i$ which is the conjunction of $\pi^*(E_I)$ and E_i. Note that E thus is a function $\coprod_{i \in X_I} X_i \to P(U(A_I \times A))$.)

The functor $\mathcal{P}$ maps a morphism

$$(u, (f_i)_{i \in X_I})\colon (I, A, (X_i, E_i)_{i \in X_I}) \to (J, B, (Y_j, E'_j)_{j \in X_J}),$$

with $I = (X_I, A_I, E_I)$ and $J = (X_J, A_J, E_J)$, to the square

$$\begin{tikzcd}
(\coprod_{i \in X_I} X_i, A_I \times A, E) \arrow[r, "\{u,f\}"] \arrow[d, "\pi"'] & (\coprod_{j \in X_I} Y_j, A_J \times B, E') \arrow[d, "\pi"] \\
I \arrow[r, "u"'] & J
\end{tikzcd}$$

where $\{u, f\}$ is the function $(i, x) \mapsto (u(i), f_i(x))$. We are to show that $\{u, f\}$ is a well-defined morphism in $\mathbf{Asm}(p)$, *i.e.*, that

$$(i, x)\colon \textstyle\coprod_{i \in X_I} X_i \mid E(i, x) \vdash E'(u(i), f_i(x))$$

holds in the logic of p. Unwinding the definitions this amounts to showing that

$$(i, x)\colon \textstyle\coprod_{i \in X_I} X_i \mid E_I(i) \wedge E_i(x) \vdash E_J(u(i)) \wedge E'_{u(i)}(f_i(x))$$

holds in p, but this holds since $(u,(f_i)_{i\in X_I})$ is a morphism in the category $\mathbf{UFam}(\mathbf{Asm}(p))$.

One can now verify that $\mathcal{P}$ is a full and faithful fibred functor. Moreover we can define a fibred functor $Q\colon \mathbf{Asm}(p)^{\to} \to \mathbf{UFam}(\mathbf{Asm}(p))$ mapping $\varphi\colon X \to I$, with $I = (X_I, A_I, E_I)$ and $X = (X_X, A_X, E_X)$ to the family $(I, A_X, (X_i, E_i)_{i\in X_I})$ with $X_i = \varphi^{-1}(i)$ and $E_i(x) = E_X(x)$. A morphism (u,f) as in

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

is mapped by Q to $(u,(f)_{i\in X_I})$. It can then be verified that Q is also a fibred functor and that $PQ \cong id$ vertically and that $QP \cong id$ vertically. $\square$

Consider the fibration $\begin{array}{c}\mathbf{UFam}(\mathbf{Asm}(p))\\ \downarrow\\ \mathbf{Asm}(p)\end{array}$. It is easy to see that

$$(I, 1_{\mathbb{C}}, (1_{\mathbf{Set}}, \top_1)_{i\in X_I})$$

is the terminal object in the fibre over $I = (X_I, A_I, E_I)$, where $1_{\mathbb{C}}$ is the terminal object in $\mathbb{C}$ and $1_{\mathbf{Set}} = \{*\}$ is the terminal object in **Set**. The terminal object functor

$$1\colon \mathbf{Asm}(p) \to \mathbf{UFam}(\mathbf{Asm}(p))$$

maps an object $I = (X_I, A_I, E_I)$ to the terminal object over I and a morphism $u\colon I \to J$ to the morphism $(u, (\lambda x.\ *)_{i\in X_I})$. Define the functor $\{-\}$ by

$$\{-\} = \mathrm{dom}\circ\mathcal{P}\colon \mathbf{UFam}(\mathbf{Asm}(p)) \to \mathbf{Asm}(p).$$

Lemma 3.6.17. *The functor $\{-\}$ is right adjoint to the terminal object functor* 1. *Moreover,* 1 *is full and faithful.*

Proof. It is straightforward to see that, for any $I \in \mathbf{Asm}(p)$, we have $I \cong \{1(I)\}$. Thus the unit $\eta\colon id \Rightarrow \{\}\circ 1$ of the adjunction is an isomorphism, and hence the left adjoint 1 is full and faithful [Mac71, dual of Theorem 1, Page 88]. $\square$

Since $\mathcal{P}$ (defined in the proof of Proposition 3.6.16) is full and the terminal object functor has a right adjoint we get the following corollary.

Corollary 3.6.18. *The functor $\mathcal{P}\colon \mathbf{UFam}(\mathbf{Asm}(p)) \to \mathbf{Asm}(p)^{\to}$ is a split full comprehension category with unit.*

We now proceed to show that the comprehension category has split products and strong split coproducts. To this end, let $\mathcal{X} = (I, A, (X_i, E_i)_{i\in X_I})$ be a family over $I = (X_I, A_I, E_I)$ and let $\pi_{\mathcal{X}}\colon \{\mathcal{X}\} = (\coprod_{i\in X_I} X_i, A_I \times A, E) \to I$ be the associated projection. We are to show that $\pi^*_{\mathcal{X}}$ has a right adjoint $\prod_{\mathcal{X}}$, which satisfies the Beck-Chevalley condition. Define

$$\prod\nolimits_{\mathcal{X}} \left((\textstyle\coprod_{i\in X_I} X_i, A_I \times A, E), C, (Z_k, E'_k)_{k\in\coprod_{i\in X_I} X_i} \right)$$

to be

$$\left(I, W, (U_i, E''_i)_{i\in X_I}\right),$$

where $W = [1 \rightharpoonup [A \rightharpoonup C]]$ is the weak partial exponential of 1 and and $[A \rightharpoonup C]$, the weak partial exponential of A and C in $\mathbb{C}$ and where

$$U_i = \{\, f\colon X_i \to \bigcup_{x\in X_i} Z_{(i,x)} \mid \forall x \in X_i.\ f(x) \in Z_{(i,x)} \text{ and } E''_i(f) \text{ is valid in } p \,\}$$

$$f\colon U_i \mid E''_i(f) \overset{\text{def}}{=} \forall x\colon X_i.\ E_i(x) \supset E'_{(i,x)}(f(x)).$$

Remark 3.6.19. It may be useful to explicitly state how the existence predicate E''_i is interpreted in the realizability pretripos. Recall that for each $i \in X_I$, we are given a set X_i (an object in the base category) and a predicate

$$(A, E_i) \in \mathbf{UFam}(\mathbb{C})_{X_i}$$

in the fibre over X_i. We omit the A in the following. Moreover, for each $(i, x) \in \coprod_{i\in X_I} X_i$ we are given a predicate

$$(C, E'_{(i,x)}) \in \mathbf{UFam}(\mathbb{C})_{Z_{(i,x)}}$$

in the fibre over $Z_{(i,x)}$. We omit the C in the following. Consider the following diagram

$$\begin{tikzcd}
E_i & (\pi'^* E_i \supset a^* E'_{(i,x)}) & E'_{(i,x)} & \forall_\pi(\pi'^* E_i \supset a^* E'_{(i,x)}) \\
X_i & U_i \times X_i \arrow[l, "\pi'"'] \arrow[r, "a"] \arrow[rr, bend right, "\pi"'] & Z_{(i,x)} & U_i,
\end{tikzcd}$$

where a is the function $(f, x) \mapsto f(x)$. The bottom row is in the base category and the columns above each object in the base shows objects in the fibre over the corresponding base object. For example, E_i is above X_i because E_i is in the fibre over X_i. The predicates E_i and $E'_{(i,x)}$ are reindexed to the fibre over $U_i \times X_i$, as shown, and the implication of the resulting predicates is formed there. Then the $\forall_\pi$ functor is applied to get the resulting predicate over U_i. This is the predicate E''_i defined logically above. Note that $\pi'^* E_i$ and $a^* E'_{(i,x)}$ have the same underlying objects of realizers (A and C) as E_i and $E'_{(i,x)}$. By the definition of $\supset$ in the realizability pretripos, the underlying object of realizers for $(\pi'^* E_i \supset a^* E'_{(i,x)})$ is the weak exponential $[A \rightharpoonup C]$ of A and C. Then $W = [1 \rightharpoonup [A \rightharpoonup C]]$ is the underlying object of realizers for the resulting predicate obtained by applying $\forall_\pi$. That explains why we above use W as the object of realizers for the family $(U_i, E''_i)_{i \in X_I}$.

The action of $\prod_{\mathcal{X}}$ on a morphism $(id, (f_{(i,x)})_{(i,x) \in \coprod_{i \in X_I} X_i})$ is defined to be $(id, (g \mapsto \lambda x \in X_i.\ f_{(i,x)}(g(x)))_{i \in X_I})$, which is easily seen to be a well-defined morphism.

We now proceed to show that we have an adjunction

$$\mathbf{UFam}(\mathbf{Asm}(p))_I \underset{\prod_{\mathcal{X}}}{\overset{\pi^*_{\mathcal{X}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{UFam}(\mathbf{Asm}(p))_{\{\mathcal{X}\}},$$

that is,

$$\frac{\pi^*_{\mathcal{X}}(I, B, (Y_i, E_i)_{i \in X_I}) \longrightarrow (\{\mathcal{X}\}, C, (Z_k, E'_k)_{k \in \coprod_{i \in X_I} X_i})}{\overline{(I, B, (Y_i, E_i)_{i \in X_I}) \longrightarrow \prod_{\mathcal{X}} (\{\mathcal{X}\}, C, (Z_k, E'_k)_{k \in \coprod_{i \in X_I} X_i})}}$$

Using the definitions of $\pi^*_{\mathcal{X}}$ and $\prod_{\mathcal{X}}$ we are to show

$$\frac{(\{\mathcal{X}\}, B, (Y_i, E_i)_{(i,x) \in \coprod_{i \in X_I} X_i}) \longrightarrow (\{\mathcal{X}\}, C, (Z_k, E'_k)_{k \in \coprod_{i \in X_I} X_i})}{\overline{(I, B, (Y_i, E_i)_{i \in X_I}) \longrightarrow (I, W, (U_i, E''_i)_{i \in X_I})}}$$

Thus suppose $(id, (g_{(i,x)})_{(i,x) \in \coprod_{i \in X_I} X_i})$ is a morphism

$$(\{\mathcal{X}\}, B, (Y_i, E_i)_{(i,x) \in \coprod_{i \in X_I} X_i}) \longrightarrow (\{\mathcal{X}\}, C, (Z_k, E'_k)_{k \in \coprod_{i \in X_I} X_i}).$$

We define the transpose of this morphism to be

$$(id, (y \mapsto \lambda x \in X_I.\ g_{(i,x)}(y))_{i \in X_I}).$$

We have to show that it is a well-defined morphism from $(I, B, (Y_i, E_i)_{i \in X_I})$ to $(I, W, (U_i, E_i'')_{i \in X_I})$ in $\mathbf{UFam}(\mathbf{Asm}(p))_I$. Thus we are to show that

$$\emptyset \mid \emptyset \vdash \forall i \colon X_I.\ \forall y \colon Y_i.\ E_I(i) \supset (E_i(y) \supset E_i''(\lambda x \in X_I.\ g_{(i,x)}(y)))$$

is valid in the logic of the realizability pretripos p. Arguing in the logic of p, let $i \colon X_I$ and $y \colon Y_i$ and suppose that $E_I(i)$ and $E_i(y)$. Then for any $x \colon X_i$ such that $E_i(x)$ we have, by the assumption that $(id, (g_{(i,x)})_{(i,x) \in \coprod_{i \in X_I} X_i})$ is a morphism, that $E'_{(i,x)}(g_{(i,x)}(y))$, as required by the definition of E_i''.

For the other transpose, suppose that $(id, (h_i)_{i \in X_I})$ is a morphism from $(I, B, (Y_i, E_i)_{i \in X_I})$ to $(I, W, (U_i, E_i'')_{i \in X_I})$ in $\mathbf{UFam}(\mathbf{Asm}(p))_I$. We define its transpose to be

$$(id, (y \mapsto h_i(y)(x))_{(i,x) \in \coprod_{i \in X_I} X_i}).$$

This is a well-defined morphism because, arguing in the logic of the realizability pretripos p, for all $(i,x) \colon \coprod_{i \in X_I} X_i$, for all $y \colon Y_i$, supposing that $E(i,x)$ and $E_i(y)$ we have that $E'_{(i,x)}(h_i(y)(x))$, by the assumption that $(id, (h_i)_{i \in X_I})$ is a morphism.

It is straightforward to verify that the transposition operations are inverses and suitably natural. (The point is that the transposes are as in the family fibration $\begin{array}{c}\mathbf{Fam(Set)}\\ \downarrow \\ \mathbf{Set}\end{array}$ (see [Jac99]); all we really need to verify is that the transposes are well-defined morphisms but this we have already done.) Thus we have now shown that $\pi_{\mathcal{X}}^* \dashv \prod_{\mathcal{X}}$.

For the Beck-Chevalley condition we are to show that for a pullback

$$\begin{array}{ccc}
(\coprod_{i \in X_I} X_{u(i)}, A_I \times B, E) & \xrightarrow{\{u, id\}} & (\coprod_{j \in X_J} X_j, A_J \times B, E') \\
{\scriptstyle \pi_{\mathcal{X}}} \big\downarrow & & \big\downarrow {\scriptstyle \pi_{\mathcal{Y}}} \\
I & \xrightarrow[u]{\hspace{6em}} & J
\end{array}$$

in $\mathbf{Asm}(p)$, we have that the canonical natural transformation

$$u^* \textstyle\prod_{\mathcal{Y}} \Rightarrow \prod_{\mathcal{X}} \{u, id\}^*$$

is an *identity* (not only iso, because we claim to have *split* products). This is tedious but straightforward to verify.

For the comprehension category to have strong split coproducts (modelling dependent sums) we need, with notation as in the previous paragraph, to have left adjoints $\coprod_{\mathcal{X}}$ to $\pi_{\mathcal{X}}^*$ for projections $\pi_{\mathcal{X}}$, satisfying a Beck-Chevalley condition. Define

$$\coprod\nolimits_{\mathcal{X}} \left((\textstyle\coprod_{i\in X_I} X_i, A_I \times A, E), C, (Z_k, E'_k)_{k\in\coprod_{i\in X_I} X_i} \right)$$

to be

$$\big(I, A\times C, (\{\,(x,z) \mid x\in X_i \text{ and } z\in Z_{(i,x)}\,\}_i, E''_i)_{i\in X_I}\big),$$

where, for all $i \in X_I$,

$$(x,z)\colon \{\,(x,z) \mid x\in X_i \text{ and } z\in Z_{(i,x)}\,\}_i \mid E''_i(x,z) \overset{\text{def}}{=} E_i(x) \wedge E'_{(i,x)}(z).$$

On a morphism $(id, (f_{(i,x)})_{(i,x)\in\coprod_{i\in X_I} X_i})$ we define $\coprod_{\mathcal{X}}$ to give

$$(id, ((x,z)\mapsto (x, f_{(i,x)}(z)))_{i\in X_I}),$$

which is easily seen to be a well-defined morphism.

We leave the verification of the adjunction $\coprod_{\mathcal{X}} \dashv \pi_{\mathcal{X}}^*$ to the reader. (Again, the proof is essentially as for $\begin{array}{c}\mathbf{Fam(Set)}\\ \downarrow \\ \mathbf{Set}\end{array}$ and one just has to verify that the transposes are well-defined morphisms by arguing in the logic of the realizability pretripos p.)

Again it is straightforward to verify that the Beck-Chevalley condition holds, *i.e.*, referring to the pullback in the previous paragraph, that $\coprod_{\mathcal{X}} \{u, id\}^* \Rightarrow u^* \coprod_{\mathcal{Y}}$ is an identity. This shows then that we have split coproducts. To have *strong* split coproducts, we have to show that the canonical maps κ in

$$\begin{tikzcd}
(\coprod_{(i,x)\in\coprod_{i\in X_I} X_i} X_i, (A_I\times A)\times C, E) \arrow[r, "\kappa"] \arrow[d, "\pi"'] & (\coprod_{i\in X_I}\{\,(x,z)\mid x\in X_i, z\in Z_{(i,x)}\,\}, A_I\times(A\times C), E') \arrow[d, "\pi"] \\
(\coprod_{i\in X_I} X_i, A_I\times A, E'') \arrow[r, "\pi_{\mathcal{X}}"'] & I
\end{tikzcd}$$

is an isomorphism. But κ is just the map $((i,x),z)\mapsto (i,(x,z))$ which clearly has an inverse. Hence we have strong coproducts.

In summa, we have proved the following theorem.

Theorem 3.6.20. *The functor $\mathcal{P}\colon \mathbf{UFam(Asm(}p\mathbf{))} \to \mathbf{Asm(}p\mathbf{)}^{\to}$ is a split closed comprehension category. Hence, we have a split model of dependent type theory.*

3.6.3 Stable and Disjoint Coproducts

By Theorem 3.5.3 we know that the category of assemblies constructed over a pretripos with disjunction has finite coproducts. In this subsection we prove that if the pretripos is a weak *realizability* tripos, then the finite coproducts are stable and disjoint. Stability and disjointness are important for the interpretation of logic by means of subobjects. We begin by recalling (*e.g.*, from [CLW93]) the definition of stability and disjointness.

Definition 3.6.21. In a category with finite coproducts and pullbacks along injections, coproducts are said to be **disjoint** if for any finite sum $Y = Y_1 + \cdots + Y_n$, the pullback $Y_i \times_Y Y_j$ is isomorphic to 0 (the initial object, *i.e.*, the empty sum) whenever $i \neq j$, and all injections are monic:

$$\begin{tikzcd} 0 \cong Y_i \times_Y Y_j \arrow[r] \arrow[d] & Y_j \arrow[d, tail] \\ Y_i \arrow[r, tail] & Y. \end{tikzcd}$$

Definition 3.6.22. In a category with finite sums and pullbacks along their injections, a coproduct diagram

$$X \xrightarrow{\kappa} X + Y \xleftarrow{\kappa'} Y$$

is said to be **universal** or **stable** if pulling it back along any morphism into $X + Y$ gives a coproduct diagram.

Theorem 3.6.23. *Let $\mathbb{C}$ be a WCPC-category with weak finite coproducts and let* $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow p\\ \mathbf{Set}\end{array}$ *be the induced realizability pretripos with disjunction. Then* $\mathbf{Asm}(p)$ *has stable and disjoint finite coproducts.*

Proof. By Theorem 3.5.3 it suffices to show that the coproducts are stable and disjoint. Disjointness follows from disjointness of coproducts in **Set** and the explicit description of pullbacks and coproducts in $\mathbf{Asm}(p)$. For stability we reason as follows. Consider the following diagram in $\mathbf{Asm}(p)$.

$$\begin{tikzcd} & P \arrow[dl, "\kappa_P"'] \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & (X, A, E_X) \arrow[d, "\kappa"] \\ P + Q \arrow[r, dashed] & (Z, C, E_Z) \arrow[r, "f"] & (X + Y, V, E) \\ & Q \arrow[ul, "\kappa_Q"] \arrow[u, "\pi"] \arrow[r] \arrow[ur, phantom, "\urcorner", very near start] & (Y, B, E_Y), \arrow[u, "\kappa'"'] \end{tikzcd}$$

where P is the pullback of (X, A, E_X) along f and Q is the pullback of (Y, B, E_Y) along f and $P + Q$ is the coproduct of P and Q. We are to show that $P \longrightarrow (Z, C, E_Z) \longleftarrow Q$ is a coproduct diagram. We do this by showing that (Z, C, E_Z) is isomorphic to $P + Q$. By the universal property of $P + Q$, there is a unique morphism from $P + Q$ to (Z, C, E_Z) such that the two triangles on the left in the diagram above commute. Consider the function g from Z to the underlying set of $P + Q$ defined by

$$z \mapsto \begin{cases} \kappa_P(z, x) & \text{if } f(z) = \kappa(x) \text{ for some } x \in X \\ \kappa_Q(z, y) & \text{if } f(z) = \kappa'(y) \text{ for some } y \in Y. \end{cases}$$

(This is a well-defined function since κ and κ' are monic.) Clearly, if g is a morphism from (Z, C, E_Z) to $P + Q$ in $\mathbf{Asm}(p)$ it establishes the required isomorphism since g is the unique map in **Set** such that $g\pi = \kappa_P$ and $g\pi = \kappa_Q$. We now proceed to show that g is a well-defined morphism, that is, that

$$z : Z \mid E_Z(z) \vdash E_{P+Q}(g(z)) \tag{3.5}$$

is valid in the logic of p, where E_{P+Q} is the underlying existence predicate of $P+Q$. (Note that we here follow our convention of leaving out the object of realizers when it is clear from context, *e.g.* writing $E_Z(z)$ for $(C, E_Z)(z)$.) Using that f is a morphism we have that

$$\begin{aligned} z : Z \mid E_Z(z) \vdash \ & (\exists x : X.\ f(z) = \kappa(x) \wedge E_X(x)) \vee \\ & (\exists y : Y.\ f(z) = \kappa'(y) \wedge E_Y(y)), \end{aligned}$$

from which the required follows, using that equality is very strong in p (see Page 54). □

The following corollaries are then obtained using 3.4.10 and 3.4.14.

Corollary 3.6.24. *Let $(A, \cdot)$ be a partial combinatory algebra and let $\mathbb{C}(A)$ be the WCPC-category induced by $(A, \cdot)$. Let* $\begin{array}{c}\mathbf{UFam}(\mathbb{C}(A))\\ \downarrow p\\ \mathbf{Set}\end{array}$ *be the realizability pretripos over $\mathbb{C}(A)$. Then $\mathbf{Asm}(p)$ has stable and disjoint finite coproducts.*

Corollary 3.6.25. *Let* $\begin{array}{c}\mathbf{UFam}(\mathbf{ALat})\\ \downarrow p\\ \mathbf{Set}\end{array}$ *be the realizability pretripos over* **ALat**. *Then $\mathbf{Asm}(p)$ has stable and disjoint finite coproducts.*

3.7 Modest Sets over Realizability Pretriposes

In this section we show how we can also generalize the construction of the category of modest sets over a PCA(see, *e.g.*, [Hyl82, LM91, Lon94]) and over algebraic lattices [BBS98]. Given the work we have done with assemblies, the development in this section is quite straightforward and standard and we shall leave the verification of many details to the interested reader.

Convention 3.7.1. For the remainder of this section, let $\mathbb{C}$ be a WCPC-category and let $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow p\\ \mathbf{Set}\end{array}$ be the realizability pretripos over $\mathbb{C}$. As before, we sometimes write $\mathbf{Asm}(\mathbb{C})$ for $\mathbf{Asm}(p)$.

Definition 3.7.2. An object $(X, A, E) \in \mathbf{Asm}(p)$ is called **modest** if

$$\forall x, x' \in X.\ \big(E(x) \cap E(x') \neq \emptyset \implies x = x'\big).$$

Definition 3.7.3. The full subcategory of $\mathbf{Asm}(p)$ formed by the modest sets is referred to as the category of **modest sets over the realizability pretripos** $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow p\\ \mathbf{Set}\end{array}$ and is denoted $\mathbf{Mod}(p)$ (or $\mathbf{Mod}(\mathbb{C})$).

Remark 3.7.4. We defined a category of assemblies over any regular fibration over **Set**, but the definition of modest sets is only given for realizability pretriposes. It may be possible to suitably generalize the definition of modest sets to work over more general fibrations but the resulting definition would probably not be as concrete as the above (where we explicitly use intersection of sets of realizers) and we have thus decided to stick with this definition since it also covers all our applications.

Example 3.7.5. Let A be a PCA and let $\mathbb{C}(A)$ be the induced WCPC-category (see Definition 3.1.17). Then the category of modest sets over the realizability pretripos $\begin{array}{c}\mathbf{UFam}(\mathbb{C}(A))\\ \downarrow p\\ \mathbf{Set}\end{array}$ is equivalent to the usual category $\mathbf{Mod}(A)$ of modest sets over A as defined, *e.g.*, in [Lon94].

Example 3.7.6. Let $\begin{array}{c}\mathbf{UFam}(\mathbf{ALat})\\ \downarrow p\\ \mathbf{Set}\end{array}$ be the realizability pretripos over **ALat**. Then $\mathbf{Mod}(p)$ is equivalent to the category of modest sets over **ALat** as defined in [BBS98]. This category is equivalent to the category **Equ** of equilogical spaces mentioned in the introduction to this chapter; see [BBS98].

Just as for modest sets over PCAs there is an equivalent definition in terms of partial equivalence relations:

Definition 3.7.7. The category $\mathbf{PER}(\mathbb{C})$ of **partial equivalence relations over** $\mathbb{C}$ is the category with

objects pairs (A, R) with $A \in \mathbb{C}$ and R a partial equivalence relation on $U(A)$;

morphisms $(A, R) \to (B, S)$ are equivalence classes of morphisms

$$f: A \to B$$

that satisfy

$$\forall a, a' \in U(A).\ a\ R\ a' \implies \begin{array}{l} U(f)(a)\downarrow \text{ and } U(f)(a')\downarrow \\ \text{and } U(f)(a)\ S\ U(f)(a'); \end{array}$$

with two such f and f' equivalent iff

$$\forall a \in U(A).\ a\ R\ a \implies U(f)(a)\ S\ U(f')(a).$$

By analogy with the situation over PCAs we have the following proposition (we omit the easy proof).

Proposition 3.7.8. *The category* $\mathbf{PER}(\mathbb{C})$ *is equivalent to* $\mathbf{Mod}(p)$.

Let I denote the inclusion functor $\mathbf{Mod}(p) \to \mathbf{Asm}(p)$.

Proposition 3.7.9. $\mathbf{Mod}(p)$ *is a regular category and the inclusion* I *is exact.*

Proof. Finite limits and image factorizations are calculated as in $\mathbf{Asm}(p)$; one just need to verify that the resulting existence predicates are all modest, but that is straightforward. □

Define the functor $R\colon \mathbf{Asm}(p) \to \mathbf{Mod}(p)$ as follows. On objects (X, A, E), let $R(X, A, E) = (X/{\sim}, A, E')$, where $\sim$ is the transitive closure of $\smile$ in X with $x \smile x'$ iff $E(x) \cap E(x') \neq \emptyset$ and $E'([x]) = \bigcup_{x' \in [x]} E(x')$. On morphisms f, let $R(f)$ be the mapping $[x] \mapsto [f(x)]$.

Proposition 3.7.10. *There is an adjunction* $R \dashv I$ *and* R *preserves products. Thus* $\mathbf{Mod}(p)$ *is an exponential ideal of* $\mathbf{Asm}(p)$.

Proof. Simple verification. □

In fact, $\mathbf{Mod}(p)$ is also locally cartesian closed and we can define the category $\mathbf{UFam}(\mathbf{Mod}(p))$ of uniform modest sets in much the same way as $\mathbf{UFam}(\mathbf{Asm}(p))$ is defined (the definition has been written out for p the realizability pretripos over $\mathbf{ALat}$ in [BBS98]). Then we again get that the projection $\mathbf{UFam}(\mathbf{Mod}(p)) \to \mathbf{Mod}(p)$ is a split fibration which is equivalent as a fibration to the codomain fibration over $\mathbf{Mod}(p)$, via a functor

$$\mathcal{P}\colon \mathbf{UFam}(\mathbf{Mod}(p)) \to \mathbf{Mod}(p)^{\to}.$$

One can also show that this functor is a split closed comprehension category. Thus we also get a split model of dependent type theory over $\mathbf{Mod}(p)$.

3.8 Relation to Regular and Exact Completions

As mentioned in the introduction to this chapter, in our work with Carboni, Rosolini and Scott [BCRS98] we have developed a complementary approach to a general notion of realizability for type theory, based on the theory of exact categories and exact completions. The approach in [BCRS98], see also [CR99], is a generalization of "the exact-completion approach to realizability toposes" as found in [RR90], see also [Car95], whereas the approach in this chapter is a generalization of "the tripos-theoretic approach to realizability toposes" as found in [HJP80, Pit81].

In this section we briefly relate the approach in [BCRS98] to the approach in this chapter. For this section only, we assume that the reader has some knowledge of regular and exact completions (the material included in [BCRS98] suffices; for an in-depth treatment see [Car95, CV98]).

Convention 3.8.1. For the remainder of this section, let $\mathbb{C}$ be a WCPC-category and let $U\colon \mathbb{C} \to \mathbf{Ptl}$ be a WCPC-functor.

Recall the following definition from [CR99]. (It is slightly different from a related definition in [BCRS98], which is geared more towards situations where all realizers are *total*.)

Definition 3.8.2. The category $\mathcal{F}(\mathbb{C})$ is the category with

objects triples (X, A, σ), where $X \in \mathbf{Set}$, $A \in \mathbb{C}$, and $\sigma\colon X \to U(A)$ in $\mathbf{Set}$;

morphisms $f\colon (X, A, \sigma) \to (Y, B, \tau)$ are functions $f\colon X \to Y$ in $\mathbf{Set}$

such that there exists a $g\colon A \to B$ in $\mathbb{C}$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sigma"'] & Y \arrow[d, "\tau"] \\ U(A) \arrow[r, "U(g)"'] & U(B) \end{tikzcd}$$

commutes in **Ptl** (note that only $U(g)$ may be partial).

Remark 3.8.3. Let A be a PCA and let $\mathbb{C}(A)$ be the WCPC-category induced by A. Then $\mathcal{F}(\mathbb{C}(A))$ is equivalent to the category of *partitioned assemblies* over A, as defined in [CFS88, RR90].

The following proposition is straightforward to show (very similar results can be found in [CR99]), and we omit the proof.

Proposition 3.8.4. *Let $\mathbb{C}$ be a WCPC-category and suppose $U\colon \mathbb{C} \to$ **Ptl** is a WCPC-functor. Then $\mathcal{F}(\mathbb{C})$ has all finite limits.*

We briefly recall the explicit description of the regular completion of a lex category [Car95], which we shall use below. For further information about the regular completion of a lex category, see [Car95].

Let $\mathbb{C}$ be a lex category. Then the **regular completion of** $\mathbb{C}$ is the category $(\mathbb{C})_{\mathrm{reg/lex}}$ with

objects morphisms

$$\begin{tikzcd} U \arrow[d, "f"'] \\ V \end{tikzcd}$$

of $\mathbb{C}$;

morphisms

$$\begin{tikzcd} X \arrow[d, "f"'] & U \arrow[d, "g"] \\ Y \arrow[r, phantom, "\xrightarrow{[l]}"] & V \end{tikzcd}$$

are equivalence classes of morphisms $l\colon X \to U$ such that $glf_0 = glf_1$, where f_0, f_1 are the structural maps of the kernel of f, with two such arrows l and l' equivalent if $gl = gm$.

The following is an easy generalization of [Car95, Lemma 6.1] (which deals with the special case where the WCPC-category is the one induced by the Kleene PCA).

Proposition 3.8.5. *The category* $\mathbf{Asm}(\mathbb{C})$ *is equivalent to* $(\mathcal{F}(\mathbb{C}))_{reg/lex}$.

Proof. The proof is essentially as in [Car95, Lemma 6.1].[7] Define the functor $G\colon (\mathcal{F}(\mathbb{C}))_{\mathrm{reg/lex}} \to \mathbf{Asm}(\mathbb{C})$ as follows. An object

$$\begin{tikzcd} (X, A, \sigma) \arrow[d, "f"'] \\ (Y, B, \tau) \end{tikzcd}$$

in $(\mathcal{F}(\mathbb{C}))_{\mathrm{reg/lex}}$ is mapped to the object $(\mathrm{Im}(f), E)$, where

$$E(y) \stackrel{\mathrm{def}}{=} \{\, a \in U(A) \mid \exists x \in X.\ f(x) = y \wedge \sigma(x) = a \,\}.$$

A morphism

$$\begin{tikzcd} (X, A, \sigma) \arrow[d, "f"'] & & (X', A', \sigma') \arrow[d, "g"'] \\ (Y, B, \tau) \arrow[r, phantom, "\xrightarrow{[l]}"] & & (Y', B', \tau') \end{tikzcd}$$

is mapped by G to the function $y \mapsto g(l(x))$, where $x \in f^{-1}(y)$. By the fact that $[l]$ is a morphism in $(\mathcal{F}(\mathbb{C}))_{\mathrm{reg/lex}}$, this function is well-defined, in particular independent of the choice of representative for $[l]$. Now one can verify that G indeed is a functor and that it is full,[8] faithful and essentially surjective as in [Car95, Proof of Lemma 6.1]. □

Corollary 3.8.6. *There is an equivalence of categories* $(\mathbf{Asm}(\mathbb{C}))_{ex/reg} \simeq (\mathcal{F}(\mathbb{C}))_{ex/lex}$.

Proof. By Proposition 3.8.5 using that $(\mathcal{F}(\mathbb{C}))_{\mathrm{ex/lex}} \simeq ((\mathcal{F}(\mathbb{C}))_{\mathrm{reg/lex}})_{\mathrm{ex/reg}}$. □

[7] Carboni [Car95] uses another, but equivalent, formulation of the category of assemblies than we do. The proof here just combines Carboni's proof with the equivalence between the two equivalent definitions of the category of assemblies.

[8] To show that G is full, one uses choice in **Set**.

In [CR99, Page 13] it is shown that $\mathcal{F}(\mathbb{C})$ is *weakly locally cartesian closed* (see *loc. cit.* for a precise definition) when $\mathbb{C}$ is weakly cartesian closed. The same construction can be used to show that also $\mathcal{F}(\mathbb{C})$ is weakly locally cartesian closed when $\mathbb{C}$ is a WCPC-category. We can then conclude using [CR99] that $(\mathcal{F}(\mathbb{C}))_{\mathrm{ex/lex}}$ is locally cartesian closed. Thus, by Proposition 3.8.5 and by using properties of the exact completion of a regular category, we have that $\mathbf{Asm}(\mathbb{C})$ is a full subcategory of $(\mathcal{F}(\mathbb{C}))_{\mathrm{ex/lex}}$. Furthermore, one can show that $\mathbf{Asm}(\mathbb{C})$ is equivalent to the full subcategory of $(\mathcal{F}(\mathbb{C}))_{\mathrm{ex/lex}}$ on those pseudo-equivalence relations $X_1 \underset{r_2}{\overset{r_1}{\rightrightarrows}} X_0$ for which $\langle r_1, r_2\rangle$ is monic. It follows by [BCRS98, Theorem 4.3] that $\mathbf{Asm}(\mathbb{C})$ is a reflective subcategory with exponentials computed as in $(\mathcal{F}(\mathbb{C}))_{\mathrm{ex/lex}}$. By Proposition 3.7.10 we can then conclude that $\mathbf{Mod}(\mathbb{C})$ is also a reflective subcategory of $(\mathcal{F}(\mathbb{C}))_{\mathrm{ex/lex}}$ with exponentials in $\mathbf{Mod}(\mathbb{C})$ computed as in $(\mathcal{F}(\mathbb{C}))_{\mathrm{ex/lex}}$.

3.9 Other Related Work

In a talk in Cambridge, August 1995, Samson Abramsky [Abr95] made some observations related to the ones I have made in this chapter.[9] Abramsky's and my work were done independently and I did not hear about Abramsky's work until I had finished the work reported here. Abramsky showed that if a category $\mathbb{C}$ is cartesian closed, then the category of assemblies and modest sets over it are both locally cartesian closed. He further showed that if $\mathbb{C}$ has weak coproducts then the category of assemblies and modest sets have coproducts. Abramsky proceeded by analogy to realizability over PCA's, defining the categories of assemblies and modest sets directly without using a notion of pretripos. Abramsky's result is a special case of ours where the underlying WCPC-category $\mathbb{C}$ is in fact cartesian closed and the functor $U\colon \mathbb{C} \to \mathbf{Ptl}$ is the global sections functor followed by the inclusion of $\mathbf{Set}$ into $\mathbf{Ptl}$.

In [Lam94] Lambek describes a generalized construction of a category of PER's sending a category $\mathbb{C}$ to a category $\mathbb{C}^R$, whose objects are pairs (α, A) where A is an object of $\mathbb{C}$ and α is a family $(\alpha_C)_{C\in\mathbb{C}}$ of PER's α_C on $\mathrm{Hom}_{\mathbb{C}}(C, A)$ and whose morphisms $(\alpha, A) \to (\beta, B)$ are equivalence classes of morphisms $f\colon A \to B$ in $\mathbb{C}$ which (for all $C \in \mathbb{C}$) respect the PER's;

[9] I thank Jaap van Oosten for bringing this fact to my attention and for providing me with a copy of his notes from the talk. Also thanks to Samson Abramsky, who via email communicated the contents of his talk to me and provided me with the reference to Lambek's work described in the following.

two such f's are equivalent iff they induce the same map on the quotients. Thus compared to the definition of $\mathbf{PER}(\mathbb{C})$, the difference is that in $\mathbb{C}^R$, the PER's have arbitrary stages of definition (they are not just PER's on global elements). Lambek observes that if $\mathbb{C}$ is cartesian closed, then $\mathbb{C}^R$ is so as well, and if $\mathbb{C}$ has weak products (coproducts), then $\mathbb{C}^R$ has products (coproducts); he does not consider weak closure and he does not consider categories of partial maps.

Independently of the work reported here, John Longley has recently suggested another general framework encompassing both typed and untyped realizability [Lon99]. Longley defines a **type world** to be any non-empty set T of (names for) *types* equipped with binary operations $\times$ and $\to$ for forming product and arrow types. (There is no requirement that T be freely generated in any sense.) Longley then defines a **partial combinatory type structure** (or PCTS) over a type world T to a family of non-empty sets

$$\{ A_t \mid t \in T \}$$

together with partial "application" functions

$$\cdot_{t,u} \colon A_{t \to u} \times A_t \to A_u$$

such that, for all types t, u, and v, there exist elements

$$\begin{aligned} \mathrm{k}_{t,u} &\in A_{t \to u \to t} \\ \mathrm{s}_{t,u,v} &\in A_{(t \to u \to v) \to (t \to u) \to (t \to v)} \\ \mathrm{pair}_{t,u} &\in A_{t \to u \to t \times u} \\ \mathrm{fst}_{t,u} &\in A_{t \times u \to t} \\ \mathrm{snd}_{t,u} &\in A_{t \times u \to u} \end{aligned}$$

satisfying the following, for all appropriately types a, b, and c (we have left out the application operations)

$$\begin{gathered} \mathrm{k}ab = a \\ \mathrm{s}ab \downarrow \\ \mathrm{s}abc \geq (ac)(bc) \\ \mathrm{fst}(\mathrm{pair}ab) = a \\ \mathrm{snd}(\mathrm{pair}ab) = b, \end{gathered}$$

where $x \geq y$ means that if x is defined then y is defined and $x = y$.

For any such PCTS A, Longley then defines a category of assemblies and shows that it is regular and locally cartesian closed.

I will leave a detailed analysis of the relationship between Longley's notion of PCTS and my notion of WCPC-category and the induced notions of categories of assemblies for future work. However, I conjecture that given a PCTS we may define a category $\mathbb{C}$ with objects A_t for t in the type world and with morphisms partial PCTS-definable functions. There is then an inclusion functor $U\colon \mathbb{C} \to \mathbf{Ptl}$, and the conjecture is that $\mathrm{Split}(\mathbb{C}, U)$ is a WCPC-category $\mathbb{D}$ satisfying that the category $\mathbf{Asm}(\mathbb{D})$ of assemblies over $\mathbb{D}$ is equivalent to Longley's category of assemblies over the given PCTS.

Peter Lietz and Thomas Streicher [Lie] have, in the context of Longley's PCTS's, shown some results closely related to our Theorems 3.4.15 and 3.4.19 characterizing when the tripos-to-topos construction applied to the realizability tripos over a WCPC-category yields a topos. In particular, Lietz and Streicher show that a PCTS has a universal type iff it gives rise to a topos and iff it is equivalent to a partial combinatory algebra (in a suitable sense).

3.10 Future Work

There are many interesting questions for future work. One of the most interesting is "what is a suitable notion of morphism between WCPC-categories $\mathbb{C}$ and $\mathbb{D}$?" Such a morphism should induce morphisms between the induced realizability pretriposes over $\mathbb{C}$ and $\mathbb{D}$ (and then functors between the induced categories of assemblies over $\mathbb{C}$ and $\mathbb{D}$). The right notion should include Longley's notion of applicative transformation between partial combinatory algebras [Lon94]. We believe that a good notion of morphism from $\mathbb{C}$ to $\mathbb{D}$ would be a profunctor from $\mathbb{C}$ to $\mathbb{D}$, possibly satisfying some extra conditions. The result should be a (bi)category of WCPC-categories, generalizing Longley's category of partial combinatory algebras and applicative transformations [Lon94]. Another interesting question is what universal property $\mathbb{C}(A)$ has ?

Chapter 4

A General Notion of Realizability for Logic

In the previous chapter we saw how to obtain a model of an expressive type theory, based on our general notion of realizability. In this chapter we show that we can also obtain an expressive predicate logic to reason about the types and terms in the type theory.

We shall restrict attention to realizability over a WCPC-category $\mathbb{C}$ with weak finite coproducts. Recall from the previous Chapter 3 that the induced realizability pretripos $\begin{smallmatrix}\mathbf{UFam}(\mathbb{C})\\ \downarrow \\ \mathbf{Set}\end{smallmatrix}$ then has disjunction and that the category of assemblies $\mathbf{Asm}(\mathbb{C})$ (and also the category of modest sets $\mathbf{Mod}(\mathbb{C})$) then is regular, locally cartesian closed, and has stable and disjoint finite coproducts. It follows easily (using [Jac99, Lemma 4.5.2]) that $\mathbf{Asm}(\mathbb{C})$ is a *logos*[1] and hence, by [Jac99, Theorem 4.5.5] that the subobject fibration $\begin{smallmatrix}\mathrm{Sub}(\mathbf{Asm}(\mathbb{C}))\\ \downarrow \\ \mathbf{Asm}(\mathbb{C})\end{smallmatrix}$ is a first-order fibration.[2] This means that we have a model of predicate logic over the *simple* type theory of $\mathbf{Asm}(\mathbb{C})$. In this chapter we show how to get a model of predicate logic over the *dependent* type theory of $\mathbf{Asm}(\mathbb{C})$ (described in Section 3.6.2) so that we can have entailments of

[1] Recall, *e.g.* from [Jac99, Section 4.5], that a **logos** is a regular category with a strict initial object 0 (*i.e.*, every morphism $X \to 0$ is an isomorphism), with stable binary joins $\vee$ in each subobject poset $\mathrm{Sub}(I)$ and with right adjoints $\prod_u$ for each pullback functor $u^*\colon \mathrm{Sub}(J) \to \mathrm{Sub}(I)$.

[2] A **first-order fibration** is a regular fibration (see Section 3.2.2) which is fibred bicartesian closed and which has simple products $\prod_{(I,J)}$, see [Jac99, Section 4.2] for more details.

the form

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n \mid \varphi_1, \ldots, \varphi_m \vdash \psi,$$

where $x_1 : \sigma_1, \ldots, x_n : \sigma_n$ is a *dependent* type context and $\varphi_1, \ldots, \varphi_m$ is an ordinary proposition context. Moreover, we show how to get a model of *full subset types* and of *quotient types.* Subset types allow formation of types using predicates as in

$$\frac{\Gamma, x : \sigma \vdash \varphi : \mathsf{Prop}}{\Gamma \vdash \{\, x : \sigma \mid \varphi \,\} : \mathsf{Type}}$$

Quotient types allow formation of types using relations as in

$$\frac{\Gamma, x : \sigma, x' : \sigma \vdash R(x, x') : \mathsf{Prop}}{\Gamma \vdash \sigma/R : \mathsf{Type}}$$

where, intuitively, σ/R is the type of equivalence classes of σ generated by the least equivalence relation containing R.

In Section 4.1 we show how to get a model of dependent predicate logic. In Section 4.2 we describe dependent subset types and in Section 4.3 we cover dependent quotient types. In each of these three sections, we first recall the abstract categorical definition of a model (for dependent predicate logic, *etc.*) from [Jac99] and then prove that our particular model satisfies the abstract definition. We only describe the models for $\mathbf{Asm}(\mathbb{C})$; similar results, with the same definitions, hold for $\mathbf{Mod}(\mathbb{C})$. In Section 4.4 we present an equivalent formulation of the subobject fibration which can be used to give a very simple description of the logical operations. It is used in Appendix A, where we present a very concrete description of the calculus of dependent predicate logic and its interpretation in $\mathbf{Mod}(\mathbf{ALat})$.

Convention 4.0.1. For the remainder of this chapter let $\mathbb{C}$ be a WCPC-category with weak finite coproducts and let $U : \mathbb{C} \to \mathbf{Ptl}$ be a morphism of WCPC-categories. Write $\mathbf{Asm}(\mathbb{C})$ for the category of assemblies over the realizability pretripos $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow p\\ \mathbf{Set}\end{array}$ with disjunction induced by $\mathbb{C}$ and U. As in the previous chapter, we write (X, A, E) for an object $(X, (A, E))$ in $\mathbf{Asm}(\mathbb{C})$, *i.e.*, X is a set, A is an object of $\mathbb{C}$ and E is a function $X \to P(UA)$, such that $E(x)$ is non-empty, for all $x \in X$.

Before embarking on the dependent predicate logic, let us mention the following fact.

Fact 4.0.2. *The regular subobject fibration* $\begin{smallmatrix}\mathrm{RegSub}(\mathbf{Asm}(\mathbb{C}))\\ \downarrow\\ \mathbf{Asm}(\mathbb{C})\end{smallmatrix}$ *is a split higher-order fibration.*[3] *Its logic is classical.*

This fact is a straightforward generalization of [Jac99, Proposition 5.3.9]; it is based on the observation that the regular subobjects of an object $(X, A, E) \in \mathbf{Asm}(\mathbb{C})$ are in one-to-one correspondence with the powerset of X (this follows easily from the description of equalizers in $\mathbf{Asm}(\mathbb{C})$, see the proof of Proposition 3.3.3; see also [Jac99, Lemma 5.3.8]). The split generic object is $\nabla(2)$ (with ∇ defined in 3.3.4) — this object is not modest and thus the regular subobject fibration on $\mathbf{Mod}(\mathbb{C})$ only forms a first-order fibration. The logic is classical because predicates on (X, A, E) are modelled by subsets of X with the connectives interpreted by the boolean algebra operations on PX.

In the dependent predicate logic described in the following, we model predicates by arbitrary subobjects and not regular subobjects, for the usual reason that the intuitionistic logic is the most expressive (allows to reason about a wider collection of subobjects). Indeed, the regular subobjects are then exactly the double-negation closed subobjects.

4.1 Dependent Predicate Logic

We recall the definition of a DPL-structure from [Jac99, Section 11.2].

Definition 4.1.1. Consider the following diagram

$$\begin{tikzcd} \mathbb{D} \arrow[dr, "q"'] & \mathbb{E} \arrow[r, "\mathcal{P}"] \arrow[d, "p"] & \mathbb{B}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbb{B} & \end{tikzcd}$$

where $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{smallmatrix}$ is a preorder fibration and $\mathcal{P}\colon \mathbb{E} \to \mathbb{B}^{\to}$ is a comprehension category. Suppose this structure satisfies the following conditions

1. $\mathcal{P}$ is a closed comprehension category (for type formers $\prod$, $\coprod$, and 1);

[3] A **split higher-order fibration** $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is a split first-order fibration with $\mathbb{B}$ cartesian closed and with a split generic object, where a **split generic object** of p is an object $\Omega \in \mathbb{B}$ together with a collection of isomorphisms $\theta_I : \mathbb{B}(I, \Omega) \to \mathrm{Obj}\,\mathbb{E}_I$ natural in I; that is $\theta_J(u \circ v) = v^*(\theta_I(u))$ for $v : J \to I$. See [Jac99, Chapter 5] for details.

2. q is a fibred bicartesian closed preorder fibration (for $\top$, $\wedge$, $\bot$, $\vee$, $\supset$);

3. q has $\mathcal{P}$-products $\forall$, $\mathcal{P}$-coproducts $\exists$, and $\mathcal{P}$-equality Eq.

We then say that the structure is a **DPL-structure.**

Remark 4.1.2. Our definition differs from [Jac99, Definition 11.2.1] in that we have left out the requirement of a generic object (since we shall not be dealing with higher-order logic in this chapter).

Theorem 4.1.3. *The structure*

$$\begin{tikzcd} \mathrm{Sub}(\mathbf{Asm}(\mathbb{C})) \arrow[dr] & \mathbf{UFam}(\mathbf{Asm}(\mathbb{C})) \arrow[r, "\mathcal{P}"] \arrow[d] & \mathbf{Asm}(\mathbb{C})^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbf{Asm}(\mathbb{C}) & \end{tikzcd}$$

forms a split DPL-structure.

Proof. Condition 1 in the definition of DPL-structure is met by Theorem 3.6.20; condition 2 holds by the remarks in the introduction to this chapter; and condition 3 holds since $\mathbf{Asm}(\mathbb{C})$ is regular (by Proposition 3.3.3) and locally cartesian closed (by Theorem 3.5.1). Regularity entails that the subobject fibration has coproducts along *all* maps in the base category [Jac99, Theorem 4.4.4]; this gives the required $\mathcal{P}$-coproducts and also $\mathcal{P}$-equality Eq. Local cartesian closure entails that the subobject fibration has products along *all* maps in the base category [Jac99, Theorem 4.5.5]; in particular we have the required $\mathcal{P}$-products. □

4.2 Dependent Subset Types

We recall the definition of dependent subset types from [Jac99, Section 11.2]. Consider a DPL-structure

$$\begin{tikzcd} \mathbb{D} \arrow[dr, "q"'] & \mathbb{E} \arrow[r, "\mathcal{P}"] \arrow[d, "p"] & \mathbb{B}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbb{B} & \end{tikzcd}$$

as defined in 4.1.1. Then we can form the diagram

$$\begin{tikzcd}
\mathrm{Fam}_{\mathcal{P}}(\mathbb{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=90, "\mathrm{Fam}_{\mathcal{P}}(q)"'] & \mathbb{D} \arrow[d, "q"] \\
\mathbb{E} \arrow[r, "\{-\}=\mathrm{dom}\circ\mathcal{P}"'] \arrow[d, "p"] & \mathbb{B} \\
\mathbb{B} &
\end{tikzcd}$$

where $\mathrm{Fam}_{\mathcal{P}}(\mathbb{D})$ is defined by change-of-base and $\mathrm{Fam}_{\mathcal{P}}(q)$ is defined to be the composite $\mathrm{Fam}_{\mathcal{P}}(\mathbb{D}) \to \mathbb{E} \to \mathbb{B}$. The fibred terminal object functor $\top \colon \mathbb{E} \to \mathrm{Fam}_{\mathcal{P}}(\mathbb{D})$ is induced by the terminal object functor $\top \colon \mathbb{B} \to \mathbb{D}$ to q, namely as $X \mapsto (X, \top(\{X\}))$. In this situation, we say the DPL-structure has **(dependent) subset types** if there is a *fibred* right adjoint $\{-\}$ to $\top$ in the situation:

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "\top"'] \arrow[dr, "p"'] & & \mathrm{Fam}_{\mathcal{P}}(\mathbb{D}) \arrow[ll, bend right, "\{-\}"'] \arrow[dl, "\mathrm{Fam}_{\mathcal{P}}(q)"] \\
& \mathbb{B} &
\end{tikzcd}$$

(Note that we overload the notation and use $\{-\}$ for two distinct functors $\mathrm{Fam}_{\mathcal{P}}(\mathbb{D}) \to \mathbb{E}$ and $\mathbb{E} \to \mathbb{B}$.) Such an adjoint induces a (faithful) fibred subset projection functor $\mathrm{Fam}_{\mathcal{P}}(\mathbb{D}) \to \mathbf{V}(\mathbb{E})$ over $\mathbb{E}$, where $\mathbf{V}(\mathbb{E}) \hookrightarrow \mathbb{E}^{\to}$ is the full subcategory of vertical maps (with respect to p). That is, objects of $\mathbf{V}(\mathbb{E})$ are objects of $\mathbb{E}^{\to}$ (arrows of $\mathbb{E}$) which by p are mapped to the identity in $\mathbb{B}$. The fibred subset projection functor is given as follows. For an object $(X, \varphi) \in \mathrm{Fam}_{\mathcal{P}}$, the counit $\epsilon_{(X,\varphi)} \colon \top\{(X, \varphi)\} \to (X, \varphi)$ induces a morphism $\pi(\epsilon_{(X,\varphi)}) \colon \{(X, \varphi)\} \to X$ in $\mathbb{E}$ (where π is the functor $\mathrm{Fam}_{\mathcal{P}}(\mathbb{D}) \to \mathbb{E}$). The assignment $(X, \varphi) \mapsto \pi(\epsilon_{(X,\varphi)})$ extends to a functor $\mathrm{Fam}_{\mathcal{P}}(\mathbb{D}) \to \mathbf{V}(\mathbb{E})$ over $\mathbb{E}$, the fibred subset projection functor. We then say that we have **full** dependent subset types if the fibred subset projection functor is full. See [Jac99, Sections 4.6 and 11.2] for more details. See Page 258 for the logical rule for *full* subset types.

We can now prove that we indeed have full dependent subset types.

Theorem 4.2.1. *The DPL-structure*

$$\begin{tikzcd}
\mathrm{Sub}(\mathbf{Asm}(\mathbb{C})) \arrow[dr] & \mathbf{UFam}(\mathbf{Asm}(\mathbb{C})) \arrow[r, "\mathcal{P}"] \arrow[d] & \mathbf{Asm}(\mathbb{C})^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathbf{Asm}(\mathbb{C}) &
\end{tikzcd}$$

has full dependent subset types.

Proof. The proof is essentially as the proof for toposes in [Jac99, Proposition 11.2.4], combined with the proof of Proposition 3.6.16.

Consider the following diagram

$$\begin{tikzcd}
\mathrm{Fam}_{\mathcal{P}}(\mathrm{Sub}(\mathbf{Asm}(\mathbb{C}))) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=60, "\mathrm{Fam}_{\mathcal{P}}(q)=p\circ\pi'"'] & \mathrm{Sub}(\mathbf{Asm}(\mathbb{C})) \arrow[d, "q"] \\
\mathbf{UFam}(\mathbf{Asm}(\mathbb{C})) \arrow[r, "\{-\}=\mathrm{dom}\circ\mathcal{P}"'] \arrow[d, "p=\mathrm{cod}\circ\mathcal{P}"] & \mathbf{Asm}(\mathbb{C}) \\
\mathbf{Asm}(\mathbb{C}). &
\end{tikzcd}$$

We are to show that functor $\top$ has a fibred right adjoint $\{-\}$ as in the situation

$$\begin{tikzcd}
\mathbf{UFam}(\mathbf{Asm}(\mathbb{C})) \arrow[rr, "\top"'] \arrow[dr, "p"'] & & \mathrm{Fam}_{\mathcal{P}}(\mathrm{Sub}(\mathbf{Asm}(\mathbb{C}))) \arrow[ll, bend right=15, "\{-\}"'] \arrow[dl, "\mathrm{Fam}_{\mathcal{P}}(q)"] \\
& \mathbf{Asm}(\mathbb{C}), &
\end{tikzcd}$$

where, for an object $\mathcal{X} = (I, A, (X_i, E_i)_{i\in X_I})$ with $I = (X_I, A_I, E_I)$, $\top(\mathcal{X}) = (\mathcal{X}, \top_{\{\mathcal{X}\}})$. Define the functor $\{-\}$ as follows. An object $(\mathcal{X}, K \overset{\varphi}{\rightarrowtail} \{\mathcal{X}\})$ with $\mathcal{X}$ as before and with $K = (X_K, A_K, E_K)$ is mapped to

$$(I, A_K, (Z_i, E_i')_{i\in X_I}),$$

where $Z_i \overset{\mathrm{def}}{=} \{\, x \in X_K \mid \pi(\varphi(x)) = i \,\}$ and $E_i'(x) \overset{\mathrm{def}}{=} E_K(x)$. On a morphism from $(\mathcal{X}, K \overset{\varphi}{\rightarrowtail} \{\mathcal{X}\})$ to $(\mathcal{Y}, L \overset{\psi}{\rightarrowtail} \{\mathcal{Y}\})$ we let

$$\{((u, (f_i)_{i\in X_I}), (g_1, g_2))\} = (u, (g_1)_{i\in I})$$

(which is well-defined because, by definition of $\mathrm{Fam}_{\mathcal{P}}(\mathrm{Sub}(\mathbf{Asm}(\mathbb{C})))$, we have that $(g_1, g_2)\colon \varphi \to \psi$ in $\mathbf{Asm}(\mathbb{C})^{\to}$.) A tedious verification, which we omit here, then shows that $\{-\}$ is a *fibred* functor, *i.e.*, that it preserves cartesian morphisms (one uses that a morphism in $\big((u, (f_i)_{i\in X_I}), (g_1, g_2)\big)$ is cartesian in $\mathrm{Fam}_{\mathcal{P}}(\mathrm{Sub}(\mathbf{Asm}(\mathbb{C})))$ iff (g_1, g_2) is cartesian in $\begin{array}{c}\mathrm{Sub}(\mathbf{Asm}(\mathbb{C}))\\ \downarrow\\ \mathbf{Asm}(\mathbb{C})\end{array}$ over $\{(u, (f_i)_{i\in X_I})\}$.)

In the fibre over I, the adjoint equivalence

$$\begin{array}{cl}\top(\mathcal{X}) = (\mathcal{X}, \top_{\{\mathcal{X}\}}) \xrightarrow{\ f\ } (\mathcal{Y}, \varphi\colon K \to \{\mathcal{Y}\}) & \text{in } \mathrm{Fam}_{\mathcal{P}}(\mathrm{Sub}(\mathbf{Asm}(\mathbb{C})))_I \\ \hline\hline \mathcal{X} \xrightarrow[h]{} \{(\mathcal{Y}, \varphi\colon K \to \{\mathcal{Y}\})\} & \text{in } \mathbf{UFam}(\mathbf{Asm}(\mathbb{C}))_I\end{array}$$

is given as follows. Suppose that

$$\mathcal{X} = (I, A, (X_i, E_i)_{i\in X_I}) \quad \text{and} \quad \mathcal{Y} = (I, B, (Y_i, E_i')_{i\in X_I}).$$

For $f = \big((id, (f_i)_{i\in X_I}), (g_1, g_2)\big)$, where necessarily (by definition of category $\mathrm{Fam}_{\mathcal{P}}(\mathrm{Sub}(\mathbf{Asm}(\mathbb{C}))))$

$$g_2 = \{(id, (f_i)_{i\in X_I})\} = (i, x) \mapsto (i, f_i(x)),$$

the adjoint transpose $\hat{f}$ is $(id, (x \mapsto g_1(i, x))_{i\in X_I})$. For $h = (id, (h_i)_{i\in X_I})$, the adjoint transpose $\check{h}$ is

$$\Big(\big(id, (x \mapsto \pi'(\varphi(h_i(x))))_{i\in X_I}\big), \big((x, i) \mapsto (h_i(x)), (x, i) \mapsto \varphi(h_i(x))\big)\Big).$$

We omit the straightforward verification that the adjoint transposes are well-defined morphisms.

The unit η of the adjunction is given by

$$\eta_{\mathcal{X}}\colon \{\top(\mathcal{X})\} \to \mathcal{X} = (id, (x \mapsto (i, x))_{i\in X_I})$$

(note that $\{\top(\mathcal{X})\} = (I, A_I \times A, (\{i\} \times X_i, E_i')_{i\in X_I})$, when $\mathcal{X}$ is as given above.)

The counit ϵ of the adjunction is given by

$$\begin{aligned}\epsilon &: \top\{(\mathcal{X}, K \xrightarrow{\varphi} \mathcal{X})\} \to (\mathcal{X}, K \xrightarrow{\varphi} \mathcal{X}) \\ &= \Big(\big(id, (x \mapsto \pi'(\varphi(x)))_{i\in X_I}\big), \big((i, x) \mapsto x, (i, x) \mapsto (i, \pi'(\varphi(x)))\big)\Big).\end{aligned}$$

The remaining details of the verification that this establishes a *fibred* adjunction are left to the reader.

To show that we have *full* dependent subset types, note that the induced fibred subset projection functor maps $(\mathcal{X}, K \xrightarrow{\varphi} \mathcal{X})$ to the object $\{(\mathcal{X}, K \xrightarrow{\varphi} \mathcal{X})\} \to \mathcal{X}$ in $\mathbf{V}(\mathbf{UFam}(\mathbf{Asm}(\mathbb{C})))$ which is the morphism

$$(id, (x \mapsto \pi'(\varphi(x)))_{i \in X_I})$$

in $\mathbf{UFam}(\mathbf{Asm}(\mathbb{C}))$. Unravelling the definitions, one easily sees that the functor is full, as required. This completes the proof of the theorem. $\square$

4.3 Dependent Quotient Types

We recall the definition of dependent quotient types from Section 11.2 in [Jac99]. Consider a DPL-structure

$$\begin{tikzcd} \mathbb{D} \arrow[dr, "q"'] & \mathbb{E} \arrow[r, "\mathcal{P}"] \arrow[d, "p"] & \mathbb{B}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbb{B} & \end{tikzcd}$$

as defined in 4.1.1. Write $\{\!\{-\}\!\}$ for $\mathrm{cod} \circ \delta$, where $\delta(X)$ is the diagonal map used in Definition 3.6.2 to define equality. Thus $\{\!\{-\}\!\}$ maps an object $X \in \mathbb{E}$ to

$$\begin{tikzcd} \{\!\{X\}\!\} = \{(\mathcal{P}X)^*(X)\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{X\} \arrow[d, "\mathcal{P}X"] \\ \{X\} \arrow[r, "\mathcal{P}X"'] & pX. \end{tikzcd}$$

Type theoretically, $\{\!\{-\}\!\}$ maps a dependent type $\Gamma \vdash \sigma\colon \mathsf{Type}$ to the context $(\Gamma, x\colon \sigma, x'\colon \sigma)$ that extends Γ with two variables of type σ. Now consider the category of relations, obtained by change-of-base as in

$$\begin{tikzcd} \mathrm{RFam}_{\mathcal{P}}(\mathbb{D}) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=60, "\mathrm{RFam}_{\mathcal{P}}(q)"'] & \mathbb{D} \arrow[d, "q"] \\ \mathbb{E} \arrow[r, "\{\!\{-\}\!\}=\mathrm{cod}\circ\delta"'] \arrow[u, bend left, "\mathrm{Eq}"] \arrow[d, "p"] & \mathbb{B} \\ \mathbb{B}, & \end{tikzcd}$$

where $\mathrm{RFam}_{\mathcal{P}}(q)$ is the composite $\mathrm{RFam}_{\mathcal{P}}(\mathbb{D}) \to \mathbb{E} \to \mathbb{B}$. The fibred equality object functor $\mathrm{Eq}\colon \mathbb{E} \to \mathrm{RFam}_{\mathcal{P}}(\mathbb{D})$ is induced by the equality Eq in q with respect to $\mathcal{P}$, namely as $X \mapsto (X, \mathrm{Eq}(\top\{X\}))$. We say that the DPL-structure has **dependent quotient types** if there is a *fibred* left adjoint Q to Eq in the situation:

$$\begin{tikzcd} \mathrm{RFam}_{\mathcal{P}}(\mathbb{D}) \arrow[rr, bend left, "Q"] \arrow[dr, "\mathrm{RFam}_{\mathcal{P}}(q)"'] & & \mathbb{E} \arrow[ll, "\mathrm{Eq}"] \arrow[dl, "p"] \\ & \mathbb{B}. & \end{tikzcd}$$

Such an adjoint induces a "canonical quotient map" functor $\mathrm{RFam}_{\mathcal{P}}(\mathbb{D}) \to \mathbf{V}(\mathbb{E})$ commuting with the domain functor $\mathrm{dom}\colon \mathbf{V}(\mathbb{E}) \to \mathbb{E}$, where as in the previous section $\mathbf{V}(\mathbb{E}) \hookrightarrow \mathbb{E}^{\to}$ is the full subcategory of vertical maps (with respect to p). The canonical quotient map functor is given as follows. For an object $(X, R) \in \mathrm{RFam}_{\mathcal{P}}(\mathbb{D})$, the unit $\eta_{(X,R)}\colon (X, R) \to \mathrm{Eq}Q(X, R)$ induces a morphism $\pi(\eta_{(X,R)})\colon X \to Q(X, R)$ (where π is the functor $\mathrm{RFam}_{\mathcal{P}}(\mathbb{D}) \to \mathbb{E}$). The assignment $(X, R) \mapsto \pi(\eta_{(X,R)})$ extends to a functor $\mathrm{RFam}_{\mathcal{P}}(\mathbb{D}) \to \mathbf{V}(\mathbb{E})$ over $\mathbb{E}$, the canonical quotient functor, which commutes with the domain functor $\mathrm{dom}\colon \mathbf{V}(\mathbb{E}) \to \mathbb{E}$. We shall say that we have **full** or **effective** dependent quotient types if this functor is full and faithful, when restricted to equivalence relations. See [Jac99, Sections 4.8 and 11.2] for more details.

We can now prove that we have quotient types.

Theorem 4.3.1. *The DPL-structure*

$$\begin{tikzcd} \mathrm{Sub}(\mathbf{Asm}(\mathbb{C})) \arrow[dr] & \mathbf{UFam}(\mathbf{Asm}(\mathbb{C})) \arrow[r, "\mathcal{P}"] \arrow[d] & \mathbf{Asm}(\mathbb{C})^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbf{Asm}(\mathbb{C}) & \end{tikzcd}$$

has quotient types.

Proof. Consider the following diagram, in which RFam is defined by change-of-base,

$$\begin{tikzcd}
\mathrm{RFam} \arrow[r] \arrow[d] \arrow[dd, bend right=80, "\mathrm{RFam}(q)"'] & \mathrm{Sub}(\mathbf{Asm}(\mathbb{C})) \arrow[d, "q"] \\
\mathbf{UFam}(\mathbf{Asm}(\mathbb{C})) \arrow[r, "\{\{-\}\}=\mathrm{cod}\circ\delta"'] \arrow[u, bend left, "\mathrm{Eq}"] \arrow[d, "p"] & \mathbf{Asm}(\mathbb{C}) \\
\mathbf{Asm}(\mathbb{C}) &
\end{tikzcd}$$

and where $\mathrm{RFam}_{(q)}$ is the composite $\mathrm{RFam} \to \mathbf{UFam}(\mathbf{Asm}(\mathbb{C})) \to \mathbf{Asm}(\mathbb{C})$. For an object $\mathcal{X} = (I, A, (X_i, E_I)_{i \in X_I})$ in $\mathbf{UFam}(\mathbf{Asm}(\mathbb{C}))$, where $I = (X_I, A_I, E_I)$, functor $\{\{-\}\}$ acts as follows:

$$\{\{\mathcal{X}\}\} = \big(\textstyle\coprod_{(i,x) \in \coprod_{i \in X_I} X_i} X_i, (A_I \times A) \times A, E\big),$$

where $E((i,x),x') = E_I(i) \wedge E_i(x) \wedge E_i(x')$. The action of the functor on morphisms is the obvious one.

The fibred equality functor Eq is defined as in the definition of dependent quotient types. Let us work out explicitly what the action of Eq is on objects.

$$\begin{aligned}
\mathrm{Eq}(\mathcal{X}) &= \big(\mathcal{X}, \mathrm{Eq}(\top\{\mathcal{X}\})\big) && \text{by definition of Eq} \\
&= \big(\mathcal{X}, \mathrm{Eq}(\{\mathcal{X}\} \overset{id}{\rightarrowtail} \{\mathcal{X}\})\big) && \text{by definition of } \top \\
&= \big(\mathcal{X}, \textstyle\coprod_{\delta(\mathcal{X})}(\{\mathcal{X}\} \overset{id}{\rightarrowtail} \{\mathcal{X}\})\big) && \text{by definition of Eq, see Thm. 4.1.3} \\
&= \big(\mathcal{X}, \mathrm{Im}(\{\mathcal{X}\} \overset{\delta(\mathcal{X})}{\rightarrowtail} \{\{\mathcal{X}\}\})\big) && \text{by [Jac99, Theorem 4.4.4]} \\
&\cong \Big(\mathcal{X}, (\{\, ((i,x),x') \in \textstyle\coprod_{(i,x) \in \coprod_{i \in X_I} X_i} X_i \mid x = x' \,\}, A_I \times A, E)\Big) \\
& && \text{by proof of Prop. 3.3.3,}
\end{aligned}$$

where $E((i,x),x') = E(i,x) = E_I(i) \wedge E_i(x)$.

We are to show that Eq has a fibred left adjoint Q, as in

$$\begin{tikzcd}
\mathrm{RFam} \arrow[rr, bend left, "Q"] \arrow[dr, "\mathrm{RFam}(q)"'] & & \mathbf{UFam}(\mathbf{Asm}(\mathbb{C})) \arrow[ll, "\mathrm{Eq}"] \arrow[dl, "p"] \\
& \mathbf{Asm}(\mathbb{C}). &
\end{tikzcd}$$

For each $I \in \mathbf{Asm}(\mathbb{C})$, we define functor $Q_I \colon \mathrm{RFam}_I \to \mathbf{UFam}(\mathbf{Asm}(\mathbb{C}))_I$ among the fibres over I; we shall omit the straightforward verification that this defines a fibred functor Q. For $(\mathcal{X}, R) \in \mathrm{RFam}_I$, with $\mathcal{X}$ as above and with $R \overset{m}{\rightarrowtail} \{\{\mathcal{X}\}\}$ and $R = (X_R, A_R, E_R)$, define

$$Q_I(\mathcal{X}, R) = (I, A, (X_i/{\approx_i}, E_i')_{i \in X_I}),$$

where $\approx_i$ is the least equivalence relation on X_i (this least equivalence relation exists since $X_i \in \mathbf{Set}$) containing $\sim_i$ with

$$x \sim_i x' \iff \text{there exists an } r \in X_R \text{ such that } m(r) = ((i, x), x')$$

and where $E_i'([x]) = \bigcup_{x' \in [x]} E_i(x')$. On a morphism

$$\big((id, (f_i)_{i \in X_I}), (g_1, g_2)\big) \colon (\mathcal{X}, R) \to (\mathcal{Y}, S)$$

in RFam_I, with $\mathcal{Y} = (I, B, (Y_i, E_i'')_{i \in X_I})$ (and, necessarily, g_2 equal to $((i, x), x') \mapsto ((i, f_i(x)), f_i(x')))$ we define Q_I to give

$$\big(id, ([x] \mapsto [f_i(x)])_{i \in X_I}\big).$$

In the fibre over I, the adjoint correspondence

$$\frac{Q_I(\mathcal{X}, R) \xrightarrow{\;f\;} \mathcal{Y}}{(\mathcal{X}, R) \xrightarrow[h]{} \mathrm{Eq}(\mathcal{Y})}$$

is given as follows. If $f = (id, (f_i)_{i \in X_I})$, let

$$\hat{f} = \big((id, (x \mapsto f_i[x])_{i \in X_I}), (g_1, g_2)\big),$$

where $g_2((i, x), x') = ((i, f_i[x]), f_i[x'])$ and $g_1(r) = g_2(m(r))$. If

$$h = \big((id, (h_i)_{i \in X_I}), (g_1, g_2)\big),$$

let

$$\check{h} = (id, ([x] \mapsto h_i(x))_{i \in X_I})$$

(this is easily seen to be well-defined, in particular independent of the choice of representative of $[x]$). Using these definitions one can verify that Q is a fibred left adjoint to Eq, as required.

The canonical quotient map $\mathcal{X} \to Q(\mathcal{X}, R)$ is $(id, (x \mapsto [x])_{i \in X_I})$. □

Remark 4.3.2. Our model does not support so-called *full* or *effective* quotients, that is, we cannot show that the following rule is sound

$$\frac{\begin{array}{c}\Gamma, x\colon \sigma, x'\colon \sigma \vdash \text{"}R(x,x') \text{ is an equivalence relation"} \\ \Gamma \vdash M\colon \sigma \qquad \Gamma \vdash M'\colon \sigma\end{array}}{\Gamma \mid [M]_R =_{\sigma/R} [M']_R \vdash R(M,M')}$$

The reason is that, when we form the quotient $Q(\mathcal{X}, R)$ (for R an equivalence relation) we *forget* about the realizers for $r \in X_R$ (see the proof above), so from just knowing that two quotients are equal we cannot produce a realizer for the fact that representatives are related.

Indeed, by Proposition 4.8.6 in [Jac99] we know that the subobject fibration of a category $\mathbb{B}$ only has full or effective quotients in case $\mathbb{B}$ is *exact*[4] — and $\mathbf{Asm}(\mathbb{C})$ is not exact.[5] Had we instead of $\mathbf{Asm}(\mathbb{C})$ been working with $(\mathbf{Asm}(\mathbb{C}))_{\mathrm{ex/reg}}$ (see Section 3.8), then the above rule would have been available. We chose to stick with $\mathbf{Asm}(\mathbb{C})$ because it is a bit more concrete and, moreover, the results obtained for $\mathbf{Asm}(\mathbb{C})$ also apply directly for $\mathbf{Mod}(\mathbb{C})$, allowing us to conclude that we get a model of type theory and logic in the category of equilogical spaces or, equivalently, $\mathbf{Mod}(\mathbf{ALat})$. See Appendix A for a concrete treatment.

4.4 For a Concrete Description

In this section we define a fibration over $\mathbf{Asm}(\mathbb{C})$ and show that it is equivalent to the subobject fibration $\begin{array}{c}\mathrm{Sub}(\mathbf{Asm}(\mathbb{C}))\\ \downarrow \\ \mathbf{Asm}(\mathbb{C})\end{array}$. The equivalent fibration is used in Appendix A to give a concrete description of the interpretation of the dependent predicate logic.

We define a split indexed category $\Psi\colon \mathbf{Asm}(\mathbb{C})^{\mathrm{op}} \to \mathbf{Cat}$. For an object $I = (X_I, A_I, E_I)$, we let $\Psi(I)$ be the poset obtained as the partial order reflection[6] of the following preorder. The objects of the preorder are the objects in the fibre $\mathbf{UFam}(\mathbb{C})_{X_I}$ over X_I of the realizability pretripos $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow p \\ \mathbf{Set}\end{array}$.

[4] Recall that an **exact category** is a regular category in which each equivalence relation is **effective**, *i.e.*, a kernel pair of its quotient.

[5] We know that the category $\mathbf{Asm}(\mathbb{C})$ is in general not exact because the category of assemblies over a PCA is not exact (see, *e.g.*, [Jac99, Exercise 6.2.5]).

[6] The **partial order reflection** of a preorder $\leq$ is the partial order obtained by identifying objects X and Y for which $X \leq Y$ and $Y \leq X$.

The order of the preorder is defined as follows: $(B,\varphi) \leq (C,\psi)$ iff

$$\forall i\colon X_I.\ E_I(i) \supset (\varphi(i) \supset \psi(i))$$

holds in the logic of the realizability pretripos p. (We here follow the convention of leaving out the underlying object of realizers B for a predicate (B,φ) in the realizability pretripos when it is clear from context.) Note that objects in $\Psi(I)$ are equivalence classes of objects (B,φ). We do not distinguish notationally between an object (B,φ) and its equivalence class.

For a morphism $u\colon I \to J$ in $\mathbf{Asm}(\mathbb{C})$, with I as above and $J = (X_J, A_J, E_J)$, $\Psi(u)$ is the functor

$$(B,\varphi) \mapsto (B, \varphi \circ u).$$

This definition is clearly independent of the choice of representative of the equivalence class for (B,φ). Moreover, using the fact that u is a morphism in $\mathbf{Asm}(\mathbb{C})$ and thus that $\forall i\colon X_I.\ E_I(i) \supset E_J(u(i))$ is valid in the logic of p, it is easy to see that $\Psi(u)$ indeed is a functor.

We write $\begin{smallmatrix}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Asm}(\mathbb{C})\end{smallmatrix}$ for the split fibration obtained by the Grothendieck construction applied to Ψ. (Note that we use the same name $\mathbf{UFam}(\mathbb{C})$ for the total category of the fibration $\begin{smallmatrix}\mathbf{UFam}(\mathbb{C})\\ \downarrow\\ \mathbf{Asm}(\mathbb{C})\end{smallmatrix}$ as for the total category of the realizability pretripos $\begin{smallmatrix}\mathbf{UFam}(\mathbb{C})\\ \downarrow p\\ \mathbf{Set}\end{smallmatrix}$ even though the two categories are distinct. This should not cause any confusion since we shall never consider the total categories in isolation, but only as part of the two distinct fibrations.)

Proposition 4.4.1. *There is a fibred equivalence*

$$\begin{tikzcd}
\mathrm{Sub}(\mathbf{Asm}(\mathbb{C})) \arrow[rr, "\simeq"] \arrow[dr] & & \mathbf{UFam}(\mathbb{C}) \arrow[dl] \\
& \mathbf{Asm}(\mathbb{C}) &
\end{tikzcd}$$

over $\mathbf{Asm}(\mathbb{C})$.

Proof. We define fibred functors

$$F\colon \mathrm{Sub}(\mathbf{Asm}(\mathbb{C})) \to \mathbf{UFam}(\mathbb{C}) \quad \text{and} \quad G\colon \mathbf{UFam}(\mathbb{C}) \to \mathrm{Sub}(\mathbf{Asm}(\mathbb{C}))$$

over $\mathbf{Asm}(\mathbb{C})$ and show that they form a fibred equivalence.

For $I = (X_I, A_I, E_I)$, define functor $F_I\colon \mathrm{Sub}(\mathbf{Asm}(\mathbb{C}))_I \to \mathbf{UFam}(\mathbb{C})_I$ as follows. Let $F_I(\varphi\colon (Y, B, E_Y) \rightarrowtail I)$ be $(B, \varphi'\colon X_I \to P(\Gamma(B)))$ with

$$\varphi'(i) = \begin{cases} \emptyset & \text{if } i \notin \mathrm{Im}\,\varphi \\ E_Y(y) & \text{if } \varphi(y) = i \text{ for some } y \in Y. \end{cases}$$

Note that φ' is a well-defined function since φ is monic. It is easy to see that F_I is independent of the choice of representative for the subobject represented by φ. Moreover, F_I is a functor since if $\varphi \leq \psi$ in $\mathrm{Sub}(I)$, *i.e.*, if there is a morphism m such that

$$\begin{tikzcd} (Y, B, E_Y) \arrow[rr, "m"] \arrow[dr, "\varphi"'] & & (Z, C, E_Z) \arrow[dl, "\psi"] \\ & I & \end{tikzcd}$$

commutes, then $F_I(\varphi) = (B, \varphi') \leq (C, \psi') = F_I(\psi)$ in $\mathbf{UFam}(\mathbb{C})_I$ because m is a morphism in $\mathbf{Asm}(\mathbb{C})$.

With I as above, define functor G_I as follows. For a predicate $(B, \varphi) \in \mathbf{UFam}(\mathbb{C})_I$, let $G_I(B, \varphi)$ be the subobject

$$(\{\, i \in X_I \mid \varphi(i) \neq \emptyset \,\}, A_I \times B, E) \rightarrowtail I$$

represented by the identity function and where $E(i) = E_I(i) \wedge \varphi(i)$. G_I is independent of the choice of representative for (B, φ) and is easily seen to be a functor.

We show that $G_I \circ F_I = id$. Now $G_I(F_I(\varphi\colon (Y, B, E_Y) \rightarrowtail I))$ equals

$$(\{\, i \in X_I \mid \varphi(y) = i \text{ for some unique } y \,\}, A_I \times B, E)$$

with $E(i) = E_I(i) \wedge E_Y(y)$ for y the unique y such that $\varphi(y) = i$. We are to show that there is an isomorphism

$$\begin{tikzcd} (\{\, i \in X_I \mid \varphi(y) = i \text{ for some unique } y \,\}, A_I \times B, E) \arrow[r, shift left, "m"] \arrow[r, phantom, "\cong"] & (Y, B, E_Y) \arrow[l, shift left, "n"] \end{tikzcd}$$

in $\mathbf{Asm}(\mathbb{C})$. But if we let $m(i)$ be the unique y such that $\varphi(y) = i$ and let $n(y) = \varphi(y)$, then it is easy to see that m and n are well-defined morphisms and that the constitute the required isomorphism.

We next show that $F_I \circ G_I = id$. Now $F_I(G_I(B, \varphi)) = (A_I \times B, \varphi')$ with $\varphi'(i) = E_I(i) \wedge \varphi(i)$. Clearly, $(A_I \times B, \varphi') \leq (B, \varphi)$, that is, $\forall i\colon X_I.\ E_I(i) \supset$

$(\varphi'(i) \supset \varphi(i))$ holds in the logic of p. For the other direction, we have that

$$\begin{aligned} & (B, \varphi) \leq (A_I \times B, \varphi') \\ \iff & \forall i\colon X_I.\ E_I(i) \supset (\varphi(i) \supset \varphi'(i)) \quad \text{holds in the logic of } p \\ \iff & \forall i\colon X_I.\ E_I(i) \supset (\varphi(i) \supset E_I(i) \wedge \varphi(i)) \quad \text{holds in the logic of } p. \end{aligned}$$

Hence (B, φ) and $(A_I \times B, \varphi')$ represent the same object in $\mathbf{UFam}(\mathbb{C})_I$, as required. (Here we crucially use that the ordering in $\mathbf{UFam}(\mathbb{C})_I$ is defined not just as in the fibre over X_I in the realizability pretripos, but also using $E_I(i)$.)

It remains to show that the F_I's and the G_I's constitute *fibred* functors. Thus we are to show that, for any $u\colon I \to J$ in $\mathbf{Asm}(\mathbb{C})$, for all $\varphi \in \mathrm{Sub}(\mathbf{Asm}(\mathbb{C}))_J$ and all $(B, \psi) \in \mathbf{UFam}(\mathbb{C})_J$.

$$F_I(u^*(\varphi)) = u^\sharp(F_J(\varphi)) \qquad \text{and} \qquad G_I(u^\sharp(B, \psi)) = u^*(G_J(B, \psi)),$$

where we write u^* for reindexing in the subobject fibration (*i.e.*, pullback along u) and $u^\sharp$ for reindexing along u in $\begin{smallmatrix}\mathbf{UFam}(\mathbb{C}) \\ \downarrow \\ \mathbf{Asm}(\mathbb{C})\end{smallmatrix}$ (*i.e.*, composition with u). Both equalities are straightforward to show and we omit the details. □

Since the logical structure of $\begin{smallmatrix}\mathrm{Sub}(\mathbf{Asm}(\mathbb{C})) \\ \downarrow \\ \mathbf{Asm}(\mathbb{C})\end{smallmatrix}$ is defined categorically, it is preserved by the equivalence of Proposition 4.4.1. Hence we conclude by Theorem 4.1.3 that

$$\begin{tikzcd} \mathbf{UFam}(\mathbb{C}) \arrow[dr] & \mathbf{UFam}(\mathbf{Asm}(\mathbb{C})) \arrow[r, "\mathcal{P}"] \arrow[d] & \mathbf{Asm}(\mathbb{C})^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbf{Asm}(\mathbb{C}) & \end{tikzcd}$$

forms a split DPL-structure. It is not hard to show (either directly or via the equivalence in Proposition 4.4.1 and Theorem 4.1.3) that

1. The split fibred bicartesian closed structure ($\top$, $\wedge$, $\bot$, $\vee$, and $\supset$) is given exactly by the corresponding structure of the realizability pretripos.

2. The split $\mathcal{P}$-products are explicitly given as follows. Let

$$\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I}) \in \mathbf{UFam}(\mathbf{Asm}(\mathbb{C}))_I.$$

Then $\{\mathcal{X}\} = (\coprod_{i \in X_I} X_i, A_I \times A, E)$ with $E(i,x) = E_I(i) \wedge E_i(x)$. For projection $\pi_{\mathcal{X}} \colon \{\mathcal{X}\} \to I$, the right adjoint $\forall_{\mathcal{X}}$ (satisfying the Beck-Chevalley condition)

$$\begin{tikzcd} \mathbf{UFam}(\mathbb{C})_I \arrow[r, shift left, "\pi^*_{\mathcal{X}}"] \arrow[r, phantom, "\perp"] & \mathbf{UFam}(\mathbb{C})_{\mathcal{X}} \arrow[l, shift left, "\forall_{\mathcal{X}}"] \end{tikzcd},$$

is given by

$$\begin{aligned} &\forall_{\mathcal{X}}(B, \varphi \colon \textstyle\coprod_{i \in X_I} X_i \to P(UB)) \\ &\qquad\qquad = ([A \rightharpoonup B], i \mapsto \bigcap_{x \in X_i} (E_i(x) \supset \varphi(i,x))), \end{aligned}$$

where $[A \rightharpoonup B]$ is the weak partial exponential of A and B.

3. The split $\mathcal{P}$-coproducts are explicitly given as follows. Let I, $\mathcal{X}$ and $\pi^*_{\mathcal{X}}$ be as in the previous item. Then the left adjoint $\exists_{\mathcal{X}}$ (satisfying the Beck-Chevalley condition)

$$\begin{tikzcd} \mathbf{UFam}(\mathbb{C})_{\mathcal{X}} \arrow[r, shift left, "\exists_{\mathcal{X}}"] \arrow[r, phantom, "\perp"] & \mathbf{UFam}(\mathbb{C})_I \arrow[l, shift left, "\pi^*_{\mathcal{X}}"] \end{tikzcd},$$

is given by

$$\exists_{\mathcal{X}}(B, \varphi \colon \textstyle\coprod_{i \in X_I} X_i \to P(UB)) = (A \times B, i \mapsto \bigcup_{x \in X_i} (E_i(x) \wedge \varphi(i,x))).$$

4. The split $\mathcal{P}$-equality is given as follows. Let I, $\mathcal{X}$ and $\pi^*_{\mathcal{X}}$ be as in the previous items. Then

$$\{\pi^*_{\mathcal{X}}\{\mathcal{X}\}\} = (\textstyle\coprod_{(i,x) \in \coprod_{i \in X_I} X_i} X_i, (A_I \times A) \times A, E'))$$

with E' the obvious existence predicate. For the diagonal

$$\delta_{\mathcal{X}} \colon \{\mathcal{X}\} \to \{\pi^*_{\mathcal{X}}\{\mathcal{X}\}\} = (i,x) \mapsto ((i,x),x)$$

the left adjoint $\mathrm{Eq}_{\mathcal{X}}$ (satisfying the Beck-Chevalley condition)

$$\begin{tikzcd} \mathbf{UFam}(\mathbb{C})_{\mathcal{X}} \arrow[r, shift left, "\mathrm{Eq}_{\mathcal{X}}"] \arrow[r, phantom, "\perp"] & \mathbf{UFam}(\mathbb{C})_{\{\pi^*_{\mathcal{X}}\{\mathcal{X}\}\}} \arrow[l, shift left, "\delta^*_{\mathcal{X}}"] \end{tikzcd},$$

is given by

$$\mathrm{Eq}_{\mathcal{X}'}(B, \varphi)((i, x), x') = \begin{cases} \varphi(i, x) & \text{if } x = x', \\ \emptyset & \text{otherwise.} \end{cases}$$

We now show that we can get similar results for the modest sets.

Definition 4.4.2. Define the split fibration $\begin{smallmatrix}\mathbf{UFam}(\mathbb{C}) \\ \downarrow \\ \mathbf{Mod}(\mathbb{C})\end{smallmatrix}$ by change-of-base as in

$$\begin{tikzcd} \mathbf{UFam}(\mathbb{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{UFam}(\mathbb{C}) \arrow[d] \\ \mathbf{Mod}(\mathbb{C}) \arrow[r] & \mathbf{Asm}(\mathbb{C}), \end{tikzcd}$$

where the functor across the bottom is the inclusion. (Note that the category $\mathbf{UFam}(\mathbb{C})$ on the left in the diagram is different from the $\mathbf{UFam}(\mathbb{C})$ on the right.)

Proposition 4.4.3. *There is a fibred equivalence*

$$\begin{tikzcd} \mathrm{Sub}(\mathbf{Mod}(\mathbb{C})) \arrow[rr, "\simeq"] \arrow[dr] & & \mathbf{UFam}(\mathbb{C}) \arrow[dl] \\ & \mathbf{Mod}(\mathbb{C}) & \end{tikzcd}$$

over $\mathbf{Mod}(\mathbb{C})$.

Proof. The proof is as the proof of Proposition 4.4.1; the only thing to note is that the domain of the subobject $G_I(B, \varphi)$ (with G_I as in the proof of Proposition 4.4.1) indeed is a modest set. □

See Appendix A for a very concrete treatment of the interpretation of dependent predicate logic in $\mathbf{Equ} \simeq \mathbf{Mod}(\mathbf{ALat})$.

4.5 Future Work

What we have presented here is of course only the first stage, in the sense that we would like to have a wider collection of types, such as inductive, coinductive, and recursive types, and to be able to reason about those types in the predicate logic. To establish the existence of such a wider collection of types and accompanying sound reasoning principles, one would naturally assume more about the underlying WCPC-category of realizers. For instance, in the case of $\mathbf{Equ} \simeq \mathbf{Mod}(\mathbf{ALat})$ we have established, in joint work with Andrej Bauer, the existence of a wide collection of inductive and coinductive types (W- and M-types). By employing general results of Hermida and Jacobs [HJ96] we have then shown that accompanying induction and coinduction reasoning principles are sound. The details of this work will appear elsewhere.

Part II

Local Realizability Toposes and a Modal Logic for Computability

Chapter 5

Preliminaries on Tripos Theory

In this chapter we recall some of the theory of triposes which we shall make use of in the following chapters. We also include a couple of new results on triposes, see Proposition 5.4.7 and Theorem 5.4.8. The notion of tripos was invented by Hyland, Johnstone and Pitts [HJP80] and the theory of triposes (over general base categories) was developed by Pitts in his Ph.D.-thesis [Pit81]; see also [Pit99] for a retrospective survey. There is also a useful recap of the basics of tripos theory in [Jac99]. Here we only recall parts of the theory, and refer to the referenced papers for an in-depth treatment and for more examples. Apart from the new results mentioned above, our treatment is standard and closely follows the published papers, except that we present triposes as fibrations instead of indexed categories (we have chosen to do so because elsewhere we mostly use fibrations).

5.1 Definition and Examples

5.1.1 Definition and Definability Results

A tripos is a weak tripos with disjunction which has a (weak) generic object. Explicitly we define:

Definition 5.1.1. Let $\mathbb{B}$ be a finitely complete category. A **$\mathbb{B}$-tripos**, or a **tripos over** $\mathbb{B}$, is a fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ over $\mathbb{B}$ which

1. is a fibred bicartesian closed preorder (for $\top$, $\wedge$, $\supset$, $\bot$, and $\vee$); such a preorder is called a **Heyting pre-algebra** and we write $\vdash_I$ for the

preorder in the fibre $\mathbb{P}_I$;

2. has coproducts $\exists_u \dashv u^*$ along all maps $u\colon I \to J$ in the base $\mathbb{B}$;

3. has products $u^* \dashv \forall_u$ along all maps $u\colon I \to J$ in the base $\mathbb{B}$;

4. has a **weak generic predicate**: for each object I of $\mathbb{B}$ there is an object PI in $\mathbb{B}$ and an object $\in_I$ in $\mathbb{P}_{I\times PI}$, such that given any φ in $\mathbb{P}_{I\times J}$, there is a map $\{\varphi\}\colon J \to PI$ in $\mathbb{B}$ with $(id_I \times \{\varphi\})^* \in_I \dashv\vdash \varphi$ in $\mathbb{P}_{I\times J}$.

(Recall from Definition 2.2.14 that the Beck-Chevalley condition is required to hold for $\forall_u$ and $\exists_u$.)

Just as all the second-order logical connectives and quantifiers can be defined from $\supset$ and $\forall$ alone, for a tripos, the operations $\top$, $\wedge$, $\bot$, $\vee$, and $\exists$ can all be defined from $\supset$, $\forall$ and $\in$, see [HJP80, Theorem 1.4].

When the base category $\mathbb{B}$ is cartesian closed, item 4 in the definition can be replaced by:

4'. There is a **weak generic predicate** $\sigma \in \mathbb{P}_\Sigma$ over some object Σ, such that given any $\varphi \in \mathbb{P}_I$, there is a map $\{\varphi\}\colon I \to \Sigma$ in $\mathbb{B}$ with $\{\varphi\}^* \dashv\vdash_I \varphi$.

Such a weak generic predicate is called a weak generic object in [Jac99, Definition 5.2.8, Page 326].

Readers familiar [Jac99] will note that the notion of a tripos over a cartesian closed category is closely related to the notion of a higher-order fibration from [Jac99]. Besides a minor technical difference regarding Beck-Chevalley (see the footnote on Page 48), the difference is that a tripos is only required to have a *weak* generic object whereas a higher-order fibration is required to have a true generic object, *i.e.*, for a higher-order fibration the map $\{\varphi\}$ is not only required exist but also to be unique.[1]

A tripos (like a higher-order fibration) models intuitionistic higher-order logic *without* the following extensionality rule for entailment:

$$\frac{\Gamma \vdash P, Q : \sigma \to \mathsf{Prop} \quad \Gamma, x : \sigma \mid \Theta, Px \vdash Qx \quad \Gamma, x : \sigma \mid \Theta, Qx \vdash Px}{\Gamma \mid \Theta \vdash P =_{\sigma\to\mathsf{Prop}} Q}$$

For a higher-order fibration, if the fibres are not only preorders but in fact partial orders, then the fibration models the above extensionality rule (by a

[1]Thus the definition of tripos given in [Jac99, Definition 5.3.3] is in one sense more restrictive than the original definition given in [HJP80, Pit81].

simple argument analogous the proof of Theorem 5.3.7 in [Jac99]), but for a tripos this is not so because of the lack of uniqueness associated with a *weak* generic object.

A tripos $\begin{smallmatrix}\mathbb{P}\\ \downarrow \mathsf{p}\\ \mathbb{B}\end{smallmatrix}$ in which each fibre $\mathbb{P}_I$ actually is $\mathbb{B}(I, \Sigma)$, for some object $\Sigma \in \mathbb{B}$, and in which reindexing is given by composition, will be called **canonically presented**. We then write $\mathsf{p} = \mathbb{B}(-, \Sigma)$. When $\mathbb{B}$ is cartesian closed a $\mathbb{B}$-tripos can, without loss of generality, always be assumed to be canonically presented [HJP80, Pit81]. A canonically presented tripos over a topos is split iff it has *fibre-wise quantification*, see [Pit81]. For a canonically presented split $\mathcal{E}$-tripos p (for $\mathcal{E}$ a topos), with fibre over I the homset $\mathcal{E}(I, \Sigma)$, all the structure of the fibres can be defined in terms of operations on Σ. The subset $\mathsf{D}_\mathsf{p} \subseteq \mathcal{E}(1, \Sigma)$ consisting of those $\varphi \colon 1 \to \Sigma$ with $\top_1 \vdash_1 \varphi$ is called the **designated truth values**. Given the set of designated truth values, the preorder $\vdash_I$ in the fibre over I is given by

$$\varphi \vdash_I \psi \quad \text{iff} \quad \forall_I(\varphi \supset \psi) \in \mathsf{D}_\mathsf{p},$$

where I denotes the unique map $I \to 1$ in $\mathcal{E}$. So a canonically presented split $\mathcal{E}$-tripos p (for $\mathcal{E}$ a topos) may be specified by

1. an object Σ of $\mathcal{E}$;
2. maps $\supset_\mathsf{p} \colon \Sigma \times \Sigma \to \Sigma$ (for implication) and $\bigwedge_\mathsf{p} \colon (\Omega_\mathcal{E})^\Sigma \to \Sigma$ in $\mathcal{E}$ (for universal quantification);
3. a subset D_p of $\mathcal{E}(1, \Sigma)$,

satisfying various relations, see [HJP80, 1.4].

5.1.2 Topos Examples

The following example is quite trivial, but useful, *e.g.*, in connection with geometric morphisms as in Section 6.2. Let $\mathcal{E}$ be a topos. Then the subobject fibration on $\mathcal{E}$ is a tripos. It can be canonically presented as $\mathcal{E}(-, \Omega_\mathcal{E})$, where $\Omega_\mathcal{E}$ is the subobject classifier of $\mathcal{E}$.

5.1.3 Localic Examples

Let H be an internal locale in a topos $\mathcal{E}$. The **canonical $\mathcal{E}$-tripos p on H** is given by $\mathcal{E}(-, H)$, that is, the fibre over I is $\mathcal{E}(I, H)$ and reindexing is given by composition. The Heyting algebra structure on each fibre $\mathcal{E}(I, H)$

is given by the internal structure on H; quantification is given fibre-wise by the internal inf map $\bigwedge_H\colon (\Omega_{\mathcal{E}})^H \to H$; and there is just one designated truth value in D_{p}, namely the top element $\top_H\colon 1 \to H$.

Remark 5.1.2. Note that the canonical tripos on a locale is *not* obtained by viewing the locale as a WCPC-category and then doing realizability over it.

Remark 5.1.3. If $\Phi \rightarrowtail H$ is a filter on H, we can modify p by taking

$$\mathsf{D}_{\mathsf{p}} = \{\, h\colon 1 \to H \mid h \text{ factors through } \Phi \rightarrowtail H \,\}$$

and still get a tripos. More generally, if $\begin{smallmatrix}\mathbb{P}\\ \downarrow\mathsf{p}\\ \mathbb{B}\end{smallmatrix}$ is a $\mathbb{B}$-tripos and $\Phi \subseteq \mathbb{P}_1$ is a filter, we can redefine the preorder on each fibre $\mathbb{P}_I$ by

$$\varphi \vdash_I \psi \quad \text{iff} \quad \forall_I(\varphi \supset \psi) \in \Phi$$

and get a new $\mathbb{B}$-tripos, which we call p_Φ.

5.1.4 Realizability Examples

Let A be a partial combinatory algebra (PCA) in **Set**. (We here restrict attention to PCA's in **Set**, but one can also consider internal PCA's in other toposes.) We define the **standard realizability tripos** $\begin{smallmatrix}\mathbf{UFam}(A)\\ \downarrow\mathsf{p}\\ \mathbf{Set}\end{smallmatrix}$ **over** A in the following way (we write out the structure explicitly to ease some calculations later on).

As predicates on a set I one takes functions $\varphi\colon I \to PA$, ordered by

$$\varphi \vdash_I \psi \iff \exists a \in A.\ \forall i \in I.\ a \in (\varphi(i) \supset \psi(i)),$$

where, recall, for sets $X, Y \subseteq A$,

$$X \supset Y = \{\, f \in A \mid \forall a \in X.\ f \cdot a \downarrow \text{ and } f \cdot a \in Y \,\}.$$

The bicartesian closed structure for these predicates on I is given by:

$$\begin{aligned}
\top_I &= \lambda i \in I.\ A\\
\bot_I &= \lambda i \in I.\ \emptyset\\
\varphi \wedge \psi &= \lambda i \in I.\ \{\, \langle a, b\rangle \mid a \in \varphi(i) \text{ and } b \in \psi(i) \,\}\\
\varphi \vee \psi &= \lambda i \in I.\ \{\, \langle \mathbf{K}, a\rangle \mid a \in \varphi(i) \,\} \cup \{\, \langle \mathbf{KI}, b\rangle \mid b \in \varphi(i) \,\}\\
\varphi \supset \psi &= \lambda i \in I.\ \varphi(i) \supset \psi(i).
\end{aligned}$$

For $u \colon I \to J$ in **Set**, and $\varphi \colon I \to P(A)$, put

$$\forall_u(\varphi) = \lambda j \in J.\ \bigcap_{i \in I} \big((u(i) =_J j) \supset \varphi(i)\big)$$
$$\exists_u(\varphi) = \lambda j \in J.\ \bigcup_{i \in I} \{\, \varphi(i) \mid u(i) = j \,\},$$

where

$$(u(i) =_J j) = \begin{cases} A & \text{if u(i) = j} \\ \emptyset & \text{else.} \end{cases}$$

(In case $I = \emptyset$, the above intersection over I equals A.) It is easy to check that $\psi \vdash \forall_u(\varphi) \iff (\psi \circ u) \vdash \varphi$ and $\exists_u(\varphi) \vdash \psi \iff \varphi \vdash (\psi \circ u)$ and that Beck-Chevalley holds for these products and coproducts. In case u is epi, the definition of $\forall_u$ may be simplified to

$$\forall_u(\varphi) = \lambda j \in J.\ \bigcap_{i \in I} \{\, \varphi(i) \mid u(i) = j \,\}, \qquad \text{if } u \text{ is epi.}$$

The assignment $I \mapsto (PA)^I$ extends to a functor (*i.e.*, a split indexed category) $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ with reindexing by composition. The resulting fibration (obtained by the Grothendieck construction) is the tripos $\begin{array}{c}\mathbf{UFam}(A)\\ \downarrow\mathsf{p}\\ \mathbf{Set}\end{array}$ with generic predicate $id \colon (PA) \to (PA)$ over PA.

In the standard realizability tripos $\begin{array}{c}\mathbf{UFam}(A)\\ \downarrow\mathsf{p}\\ \mathbf{Set}\end{array}$ over A, the designated truth values D_{p} is the set of inhabited subsets of A.

Let $A_\sharp \subseteq A$ be a sub-PCA of A. Then, as noted in [Pit81, Page 15], taking D to be all those $1 \to PA$ corresponding to subsets $A' \rightarrowtail A$ through which some $a \colon 1 \to A$ in $A_\sharp$ factors results in a new tripos. We call this tripos **the relative realizability tripos over** A **and** $A_\sharp$ and denote it $\begin{array}{c}\mathbf{UFam}(A,A_\sharp)\\ \downarrow\mathsf{r}\\ \mathbf{Set}\end{array}$. Explicitly, predicates over I are maps $\varphi \colon I \to PA$, all the logical operations are given as for $\begin{array}{c}\mathbf{UFam}(A)\\ \downarrow\mathsf{p}\\ \mathbf{Set}\end{array}$, but the preorder is defined as

$$\varphi \vdash_I \psi \iff \exists a \in A_\sharp.\ \forall i \in I.\ a \in (\varphi(i) \supset \psi(i)),$$

that is, the required realizer has to come from the sub-PCA $A_\sharp$. We shall return to this example in the following chapter.

We also recall the following observation from [Pit81, Page 15]: For the tripos $\begin{array}{c}\mathbf{UFam}(A,A_\sharp)\\ \downarrow\mathsf{r}\\ \mathbf{Set}\end{array}$, the inhabited subsets of A form a filter

$$\Phi \subseteq \mathbf{UFam}(A, A_\sharp)_1$$

and r_Φ (see previous section) is the standard realizability tripos $\begin{array}{c}\mathbf{UFam}(A)\\ \downarrow\mathsf{p}\\ \mathbf{Set}\end{array}$ over A. We shall return to this observation in Section 6.1 in Chapter 6.

5.2 Tripos to Topos Construction

From each $\mathbb{B}$-tripos p, with $\mathbb{B}$ finitely complete, one can construct a topos, which will be denoted $\mathbb{B}[\mathsf{p}]$. The construction is a direct generalization of the construction by Higgs and Fourman-Scott [FS79] of the topos of H-valued sets of a locale H.

Definition 5.2.1. Let $\begin{array}{c}\mathbb{P}\\ \downarrow\mathsf{p}\\ \mathbb{B}\end{array}$ be a tripos. Write $\mathbb{B}[\mathsf{p}]$ for the category with

objects pairs $(I, \approx_I)$ where $I \in \mathbb{B}$ is an object of the base category and $\approx_I \in \mathbb{P}_{I\times I}$ is an "equality" predicate on I. The latter is required to be symmetric and transitive in the logic of p. This means that validity in p is required of:

$$i_1, i_2 \colon I \mid i_1 \approx_I i_2 \vdash i_2 \approx_I i_1$$
$$i_1, i_2, i_3 \colon I \mid i_1 \approx_I i_2, i_2 \approx_I i_3 \vdash i_1 \approx_I i_3.$$

morphisms $f\colon (I, \approx_I) \to (J, \approx_J)$ are equivalence classes of relations $F \in \mathbb{P}_{I\times J}$ from I to J that are

- extensional:

$$i_1, i_2 \colon I, j_1, j_2 \colon J \mid i_1 \approx_I i_2, j_1 \approx_J j_2, F(i_1, j_1) \vdash F(i_2, j_2)$$

- strict:

$$i \colon I, j \colon J \mid F(i, j) \vdash (i \approx_I i) \wedge (j \approx_J j)$$

- single-valued:

$$i \colon I, j_1, j_2 \colon J \mid F(i, j_1), F(i, j_2) \vdash j_1 \approx_J j_2$$

- total:

$$i\colon I \mid i \approx_I i \vdash \exists j\colon J.\ F(i,j)$$

The equivalence relation on these relations F is logical equivalence (in the internal language) as described by isomorphisms in the fibre. To emphasize, F and F' are related iff they are isomorphic in the fibre $\mathbb{P}_{I\times J}$. Equivalently, F and F' are related iff

$$i\colon I, j\colon J \mid \emptyset \vdash F(i,j) \supset\subset F'(i,j)$$

is valid in the logic of p. For convenience, we usually write representatives F instead of equivalence classes $[F]$. A relation F which is extensional, strict, single-valued and total, will also be called a **functional relation.**

Sometimes we omit the subscript and write $\approx$ for $\approx_I$. Further, we write $|i_1 \approx_I i_2|$ for $i_1 \approx_I i_2$ (the vertical bars are just used for bracketing to make expressions easier to read). We write $E_I(i)$ or $E(i)$ for $|i \approx_I i|$. Thus $E_I(-)$ is a unary predicate on I, defined categorically by $E_I(-) = \langle id, id\rangle^*(\approx_I) \in \mathbb{P}_I$.

The identity morphism on an object $(I, \approx_I)$ of $\mathbb{B}[\mathsf{p}]$ is the (equivalence class of the) relation $\approx_I$ itself:

$$i_1, i_2\colon I \vdash i_1 \approx_I i_2.$$

Composition of $(I,\approx_I)\xrightarrow{F}(J,\approx_J)\xrightarrow{G}(K,\approx_K)$ is the composite relation $G\circ F$,

$$i\colon I, k\colon K \vdash \exists j\colon J.\ F(i,j) \wedge G(j,k).$$

The fundamental theorem of tripos theory is:

Theorem 5.2.2. *For* $\begin{array}{c}\mathbb{P}\\ \downarrow\mathsf{p}\\ \mathbb{B}\end{array}$ *a tripos, the category* $\mathbb{B}[\mathsf{p}]$ *is a topos.*

Examples 5.2.3.

(i) Let $\mathcal{E}$ be a topos. If we apply the tripos to topos construction to the subobject fibration of $\mathcal{E}$ (the tripos from Section 5.1.2 we get, of course, back the topos $\mathcal{E}$ itself.

(ii) For a canonical tripos $\mathsf{p} = \mathcal{E}(-, H)$ on an internal locale H in a topos $\mathcal{E}$, the resulting topos $\mathcal{E}[\mathsf{p}]$ is the topos of H-valued sets [FS79], equivalent to the category of $\mathcal{E}$-valued sheaves on the locale H.

(iii) For a standard realizability tripos $\begin{array}{c}\mathbf{UFam}(A)\\ \downarrow\mathsf{p}\\ \mathbf{Set}\end{array}$ over a PCA A, we denote the resulting topos $\mathbf{Set}[\mathsf{p}]$ by $\mathrm{RT}(A)$ and refer to it as the **realizability topos over** A. In case A is K_1, $\mathrm{RT}(A)$ is the *effective topos* [Hyl82]. The topos $\mathbf{Set}[\mathsf{r}]$ obtained from the relative realizability tripos $\begin{array}{c}\mathbf{UFam}(A,A_\sharp)\\ \downarrow\mathsf{r}\\ \mathbf{Set}\end{array}$ over PCA's A and $A_\sharp$ (with $A_\sharp$ a sub-PCA of A) is denoted $\mathrm{RT}(A, A_\sharp)$ and referred to as the **relative realizability topos over** A **and** $A_\sharp$.

We now present some of the structure of $\mathbb{B}[\mathsf{p}]$ that we shall use in the sequel.

5.2.1 Finite Limits in $\mathbb{B}[\mathsf{p}]$

The terminal object in $\mathbb{B}[\mathsf{p}]$ is the terminal object $1 \in \mathbb{B}$ with equality predicate

$$x\colon 1, x'\colon 1 \vdash |x \approx_1 x'| \stackrel{\text{def}}{=} \top.$$

We shall write $1 = (1, \approx_1) \in \mathbb{B}[\mathsf{p}]$ for this object. For each object $(I, \approx_I)$ in $\mathbb{B}[\mathsf{p}]$, the (equivalence class of the) predicate $E(i) \in \mathbb{P}_I \cong \mathbb{P}_{1\times I}$ is the unique morphism from $(I, \approx_I)$ to 1.

The product of $(I, \approx_I)$ and $(J, \approx_J)$ is the object $I \times J \in \mathbb{B}$ together with equality predicate

$$z, w\colon I \times J \vdash |\pi z \approx_I \pi w| \wedge |\pi' z \approx_I \pi' w|.$$

The projection $(I, \approx) \longleftarrow (I \times J, \approx) \longrightarrow (J, \approx)$ maps are given by the predicates

$$z\colon I \times J, i\colon I \vdash |\pi z \approx_I i| \wedge E_J(\pi' z)$$
$$z\colon I \times J, j\colon J \vdash |\pi' z \approx_J j| \wedge E_I(\pi z).$$

The tupling of two maps $F\colon (K, \approx) \to (I, \approx)$ and $G\colon (K, \approx) \to (J, \approx)$ involves the predicate

$$k\colon K, z\colon I \times J \vdash F(k, \pi z) \wedge G(k, \pi' z).$$

For parallel maps F and G, an equalizer

$$(I, \approxeq) \xrightarrow{\approxeq} (I, \approx) \underset{G}{\overset{F}{\rightrightarrows}} (J, \approx)$$

is obtained by taking as new equality predicate $\approxeq$ on I,

$$i_1, i_2 \colon I \vdash |i_1 \approxeq i_2| \stackrel{\text{def}}{=} |i_1 \approx i_2| \wedge \exists j \colon J.\ F(i_1, j) \wedge G(i_2, j)$$

This predicate $\approxeq$ is also the equalizer map $\approxeq \colon (I, \approxeq) \to (I, \approx)$ of F, G.

5.2.2 Monomorphisms and Epimorphisms in $\mathbb{B}[\mathsf{p}]$

A morphism $F \colon (I, \approx) \to (J, \approx)$ in $\mathbb{B}[\mathsf{p}]$ is a monomorphism if and only if one has in the internal language of p

$$i_1, i_2 \colon I, j \colon J \mid F(i_1, j) \wedge F(i_2, j) \vdash |i_1 \approx_I i_2| \qquad \textit{i.e.}\ \text{single-valued in } i$$

Similarly, F is epi in $\mathbb{B}[\mathsf{p}]$ iff

$$j \colon J \mid \emptyset \vdash \exists i \colon I.\ F(i, j) \qquad \textit{i.e.}\ \text{total in } j$$

is valid in the logic of p.

5.2.3 Subobjects and Powerobjects in $\mathbb{B}[\mathsf{p}]$

To understand the nature of the subobjects in $\mathbb{B}[\mathsf{p}]$ one uses so-called strict predicates. For an object $(I, \approx) \in \mathbb{B}[\mathsf{p}]$, a **strict predicate** on $(I, \approx)$ is a predicate $A \in \mathbb{P}_I$ which satisfies in p

$$i_1, i_2 \colon I \mid A(i_1), i_1 \approx_I i_2 \vdash A(i_2) \qquad \text{and} \qquad i \colon I \mid A(i) \vdash E_I(i).$$

We form a category SPred(p) of $\supset\!\subset$-equivalence classes of strict predicates by stipulating that a morphism from a strict predicate A on $(I, \approx)$ to a strict predicate B on $(J, \approx)$ consists of a map $F \colon (I, \approx) \to (J, \approx)$ in $\mathbb{B}[\mathsf{p}]$ for which we have in p

$$i \colon I \mid A(i) \vdash \exists j \colon J.\ F(i, j) \wedge B(j).$$

As usual, we do not distinguish notationally between a strict predicate and its equivalence class.

The forgetful functor $\begin{smallmatrix}\mathrm{SPred}(\mathsf{p})\\ \downarrow \\ \mathbb{B}[\mathsf{p}]\end{smallmatrix}$ is a poset fibration. The order in the fibre over $(I, \approx)$ is the order inherited from p's fibre over I: for strict predicates A, B on $(I, \approx)$ one has

$$A \leq B \text{ in SPred}(\mathsf{p}) \text{ over } (I, \approx) \iff i: I \mid A(i) \vdash B(i) \text{ in } \mathsf{p}.$$

(On the left, $\leq$ is a partial order between the equivalence classes of A and B; on the right $\vdash$ is a preorder.) For a morphism $F: (I, \approx) \to (J, \approx)$ in $\mathbb{B}[\mathsf{p}]$ and a strict predicate B on $(J, \approx)$, one gets a strict predicate on $(I, \approx)$ by

$$i: I \vdash F^*(B)(i) \stackrel{\text{def}}{=} \exists j: J.\ F(i,j) \wedge B(j).$$

Strict predicates on $(I, \approx)$ correspond to subobjects of $(I, \approx)$ in $\mathbb{B}[\mathsf{p}]$ in the sense that there is an isomorphism of fibred categories,

$$\begin{tikzcd} \mathrm{Sub}(\mathbb{B}[\mathsf{p}]) \arrow[rr, "\cong"] \arrow[dr] & & \mathrm{SPred}(\mathsf{p}) \arrow[dl] \\ & \mathbb{B}[\mathsf{p}]. & \end{tikzcd}$$

Indeed, given a subobject represented by a monic $M: (Y, \approx) \rightarrowtail (X, \approx)$, we get a strict predicate on $(X, \approx)$: $x: X \vdash \exists y: Y.\ M(y, x)$. Given a strict predicate R on $(X, \approx)$ we get a new object $\|R\|$ by changing the equality on X to $x, x': X \vdash x \approx_X x' \wedge R(x)$, and then $x, x': X \vdash x \approx_X x' \wedge R(x)$ is a functional relation representing a monomorphism $\|R\| \rightarrowtail (X, \approx)$.

Given an object $(X, \approx_X)$ in $\mathbb{B}[\mathsf{p}]$, its powerobject $P(X, \approx_X)$ has underlying object PX (given by item 4 in Definition 5.1.1) and equality

$$R, S: PX \vdash |R \approx_{PX} S| \stackrel{\text{def}}{=} \forall x: X.\ (x \in_X R \supset\subset x \in_X S) \wedge E_{PX}(R)$$

where

$$\begin{aligned} E_{PX}(R) \stackrel{\text{def}}{=} & (\forall x, x': X.\ x \in_X R \wedge x \approx_X x' \supset x' \in_X R) \wedge \\ & (\forall x: X.\ x \in_X R \supset E_X(x)). \end{aligned}$$

The subobject classifier of $\mathbb{B}[\mathsf{p}]$ is the object

$$\Omega = (\Sigma, \approx_\Omega),$$

where $\Sigma \in \mathbb{B}$ is the object over which the generic predicate of p lies and $\approx_\Omega$ is logical equivalence:

$$p, q: \Sigma \vdash |p \approx_\Omega q| \stackrel{\text{def}}{=} (p \supset\subset q).$$

For a strict predicate A on $(I, \approx)$ we get a characteristic map $\mathrm{char}(A)\colon (I,\approx) \to (\Sigma,\approx)$ in $\mathbb{B}[\mathsf{p}]$ by

$$i\colon I, p\colon \Sigma \vdash \mathrm{char}(A)(i,p) \stackrel{\mathrm{def}}{=} E(i) \wedge |A(i) \supset\subset p|.$$

5.2.4 Exponentials in $\mathbb{B}[\mathsf{p}]$

In case $\mathbb{B}$ is cartesian closed, as will most often be the case in our examples, the exponentials of $\mathbb{B}[\mathsf{p}]$ can be described explicitly in the following way.

Let $\Sigma \in \mathbb{B}$ be the object over which the generic predicate in p lies. Then for the exponent of $(I,\approx)$ and $(J,\approx)$ we take $P(I \times J) = \Sigma^{I\times J}$ as the underlying object with existence predicate

$$f\colon P(I \times j) \vdash E(f) \stackrel{\mathrm{def}}{=} \text{“f is a functional relation”}.$$

That is,

$$\begin{aligned} E(f) \stackrel{\mathrm{def}}{=} & \big(\forall i_1, i_2\colon I.\ \forall j_1, j_2\colon J.\ |i_1 \approx_I i_2| \wedge |j_1 \approx_J j_2| \wedge f(i_1,j_1) \supset f(i_2,j_2)\big) \wedge \\ & \big(\forall i\colon I.\ \forall j\colon J.\ f(i,j) \supset E_I(i) \wedge E_J(j)\big) \wedge \\ & \big(\forall i\colon I.\ \forall j_1, j_2\colon J.\ f(i,j_1) \wedge f(i,j_2) \supset |j_1 \approx_J j_2|\big) \wedge \\ & \big(\forall i\colon I.\ \ E_I(i) \supset \exists j\colon J.\ f(i,j)\big). \end{aligned}$$

The equality predicate on the object $P(I \times J)$ underlying the exponent $(I,\approx) \Rightarrow (J,\approx)$ is then

$$f, g\colon P(I \times J) \vdash |f \approx g| \stackrel{\mathrm{def}}{=} E(f) \wedge E(g) \wedge \forall i\colon I.\ \forall j\colon J.\ f(i,j) \supset\subset g(i,j).$$

The evaluation map $\mathrm{Ev}\colon ((I,\approx) \times (J,\approx)) \to (J,\approx)$ is given by

$$f\colon P(I \times g), i\colon I, j\colon J \vdash \mathrm{Ev}(f,i,j) \stackrel{\mathrm{def}}{=} f(i,j) \wedge E(f).$$

For a morphism $H\colon (K,\approx) \times (I,\approx) \to (J,\approx)$, the exponential transpose $\Lambda(H)\colon (K,\approx) \to (I,\approx) \Rightarrow (J,\approx)$ is given by

$$\begin{aligned} k\colon K, f\colon I \times J \vdash \Lambda(H)(k,f) \stackrel{\mathrm{def}}{=} & E(k) \wedge E(f) \wedge \\ & \forall i\colon I.\ \forall j\colon J.\ H(f,i,j) \supset\subset f(i,j). \end{aligned}$$

5.2.5 The internal logic of $\mathbb{B}[\mathsf{p}]$

The internal logic of $\mathbb{B}[\mathsf{p}]$ (*i.e.*, the internal logic of the subobject fibration on $\mathbb{B}[\mathsf{p}]$) is most conveniently described as the internal logic of (the equivalent, see above) fibration $\begin{array}{c}\mathrm{SPred}(\mathsf{p})\\ \downarrow \\ \mathbb{B}[\mathsf{p}]\end{array}$ of strict predicates. If, for the time being we mark its connectives with a tilde $\tilde{\ }$, then expressed in terms of the connectives of p, which are written in ordinary fashion, we have

- propositional connectives in the fibre over $(I, \approx)$ are $\tilde{\bot} = \bot$, $\tilde{\vee} = \vee$, $\tilde{\top} = E_I$, $\tilde{\wedge} = \wedge$, $A \tilde{\supset} B = E_I \wedge (A \supset B)$.

- For a strict predicate A over $(I, \approx)$ and a morphism $F\colon (I, \approx) \to (J, \approx)$,

$$\tilde{\exists}_F(A)(j) = \exists i\colon I.\ F(i,j) \wedge A(i),$$
$$\tilde{\forall}_F(A)(j) = E(j) \wedge \forall i\colon I.\ F(i,j) \supset A(i).$$

  In the special case where $(I, \approx)$ is $(H, \approx) \times (J, \approx)$ and F is the projection $(H, \approx) \times (J, \approx) \to (J, \approx)$, the resulting equations are

$$\tilde{\exists} h\colon H.\ A(h,j) = \exists h\colon H.\ \ E(h) \wedge A(h,j) = \exists h\colon H.\ A(h,j),$$
$$\tilde{\forall} h\colon H.\ A(h,j) = E(j) \wedge \forall h\colon H.\ \ E(h) \supset A(h,j).$$

The generic object of $\begin{array}{c}\mathsf{SPred(p)}\\ \downarrow\\ \mathbb{B}[\mathsf{p}]\end{array}$ is the strict predicate

$$p\colon \Sigma \vdash \mathbf{true}(p) \stackrel{\text{def}}{=} |p \asymp \top|.$$

on the subobject classifier $\Omega = (\Sigma, \asymp)$

5.3 The "Constant Objects" Functor

Definition 5.3.1. Let $\mathbb{B}$ be a finitely complete category and $\begin{array}{c}\mathbb{P}\\ \downarrow \mathsf{p}\\ \mathbb{B}\end{array}$ a tripos. The **constant p-object** on an object $X \in \mathbb{B}$, denoted $\nabla_{\mathsf{p}}(X)$, has underlying object X and equality predicate $x, x'\colon X \mid (x \approx_{\nabla X} x') \stackrel{\text{def}}{=} \exists_{\delta_X}(\top_X)$, where $\delta_X : X \to X \times X$ is the diagonal map in $\mathbb{B}$. For each map $f\colon X \to Y$ in $\mathbb{B}$, $\exists_{\langle id_X, f\rangle}(\top_X) \in \mathbb{P}_{X \times Y}$ represents a map $\nabla_{\mathsf{p}}(f)\colon \nabla_{\mathsf{p}}(X) \to \nabla \mathsf{p}(Y)$. This defines a functor $\nabla_{\mathsf{p}}\colon \mathbb{B} \to \mathbb{B}[\mathsf{p}]$.

The constant objects functor is left exact.

Remark 5.3.2. The constant objects functor is so named because in localic examples it indeed assigns constant sheaves to objects from the base category. In early writings [HJP80, Pit81, Hyl82], the functor was denoted Δ_{p}; we follow the newer notational convention [CFS88, HRR90] of writing ∇_{p} for the functor because it, in realizability examples, is *right* adjoint to the global sections functor Γ and not left adjoint.

The topos $\mathbb{B}[\mathsf{p}]$ looks like being "generated by 1" over $\mathbb{B}$, in the following sense. Every object $(X, \approx)$ of $\mathbb{B}[\mathsf{p}]$ occurs as a subquotient of a constant p-object

$$\begin{array}{ccc} (X, E_X) & \rightarrowtail & \nabla_{\mathsf{p}}(X) \\ \big\downarrow & & \\ (X, \approx) & & \end{array}$$

in $\mathbb{B}[\mathsf{p}]$ (the quotient map is represented by the equality predicate on X).

5.4 Geometric Morphisms of Triposes

Geometric morphisms of triposes generalize continuous functions of locales, and just as a continuous function of locales give rise to geometric morphism of sheaves on locales, a geometric morphism of triposes gives rise to a geometric morphism of the induced toposes.

Suppose that $l \colon \mathbb{P} \to \mathbb{Q}$ is a fibred functor between triposes $\begin{smallmatrix}\mathbb{P}\\ \downarrow\mathsf{p}\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbb{Q}\\ \downarrow\mathsf{q}\\ \mathbb{B}\end{smallmatrix}$ and that l preserves fibred finite limits ($\top$ and $\wedge$). Then for an object $(X, \approx)$ in $\mathbb{B}[\mathsf{p}]$, $\bar{l}(X, \approx) = (X, l(\approx))$ is a well-defined object of $\mathbb{B}[\mathsf{q}]$. However, since l does not necessarily preserve existential quantification ($\exists$), given a functional relation $F \in \mathbb{P}_{X \times Y}$ representing a morphism $f \colon (X, \approx) \to (Y, \approx)$, $l(F) \in \mathbb{Q}_{X \times Y}$ will only be a **partial functional relation**, *i.e.*, strict, extensional, and single-valued, but not necessarily total. Using a notion of complete object we shall see how to extend l to morphisms in $\mathbb{B}[\mathsf{p}]$.

Definition 5.4.1. Let p be a tripos over $\mathbb{B}$. An object $(Y, \approx) \in \mathbb{B}[\mathsf{p}]$ is **complete** if given a partial functional relation F from $(X, \approx)$ to $(Y, \approx)$, there is $f \colon X \to Y$ in $\mathbb{B}$ such that

$$x \colon X \mid \emptyset \vdash \exists y \colon Y.\ F(x, y) \supset\subset F(x, fx)$$

is valid in the logic of p.

Remark 5.4.2. In [HJP80, Pit81] a complete object was called "weakly complete" but since it is not weak in any sense, we just the word complete.

Proposition 5.4.3. *Let* $\begin{smallmatrix}\mathbb{P}\\ \downarrow\mathsf{p}\\ \mathbb{B}\end{smallmatrix}$ *be a tripos. Any object* $(X, \approx) \in \mathbb{B}[\mathsf{p}]$ *is isomorphic to a complete one.*

Proof Sketch. $(X, \approx)$ is isomorphic to the complete object $S(X, \approx)$, where $S(X, \approx)$ is the subobject $\|S_X\| \rightarrowtail P(X, \approx_X)$ of the powerobject of $(X, \approx)$ given by the strict predicate $S_X \in \mathbb{P}_{PX}$:

$$R\colon PX \mid S_X(R) \stackrel{\text{def}}{=} \exists x\colon X.\ \big(x \approx_X x \wedge \forall x' \in X.\ (x' \in_X R \supset\subset x \approx_X x')\big).$$

See [HJP80, Pit81] for more details. $\square$

Lemma 5.4.4. *Suppose that* $l\colon \mathbb{P} \to \mathbb{Q}$ *is a fibred functor from* $\begin{array}{c}\mathbb{P}\\ \downarrow\mathsf{p}\\ \mathbb{B}\end{array}$ *to* $\begin{array}{c}\mathbb{Q}\\ \downarrow\mathsf{q}\\ \mathbb{B}\end{array}$ *preserving fibred finite limits. Let* $(X, \approx)$ *and* $(Y, \approx)$ *be objects of* $\mathbb{B}[\mathbb{P}]$ *and suppose* $(Y, \approx)$ *is complete. Then for any functional relation* F *from* $(X, \approx)$ *to* $(Y, \approx)$, $l(F)$ *is a functional relation from* $\bar{l}(X, \approx)$ *to* $\bar{l}(Y, \approx)$.

Thus given any $f = [F]\colon (X, \approx) \to (Y, \approx)$ in $\mathbb{B}[\mathsf{p}]$ with $(Y, \approx)$ complete, we can define $\bar{l}(f)$ to be $[l(F)]$. Using Proposition 5.4.3 we can then extend $\bar{l}$ to a left exact functor $\mathbb{B}[\mathsf{p}] \to \mathbb{B}[\mathsf{q}]$. For more details, see [Pit81].

Definition 5.4.5. Let $\mathbb{B}$ be a finitely complete category and let $\begin{array}{c}\mathbb{P}\\ \downarrow\mathsf{p}\\ \mathbb{B}\end{array}$ and $\begin{array}{c}\mathbb{Q}\\ \downarrow\mathsf{q}\\ \mathbb{B}\end{array}$ be triposes over $\mathbb{B}$. A **geometric morphism** $f\colon \mathsf{p} \to \mathsf{q}$ is given by a pair of fibred functors $f^*\colon \mathbb{Q} \to \mathbb{P}$ and $f_*\colon \mathbb{P} \to \mathbb{Q}$ over $\mathbb{B}$, as in

$$\begin{tikzcd}
\mathbb{P} \arrow[rr, bend left, "f_*"] \arrow[rr, phantom, "\top"] \arrow[dr, "\mathsf{p}"'] & & \mathbb{Q} \arrow[ll, bend left, "f^*"] \arrow[dl, "\mathsf{q}"] \\
& \mathbb{B} &
\end{tikzcd}$$

such that f^* is a fibred left adjoint of f_* and such that f^* preserves fibred finite limits.

Given such a geometric morphism f, since f^* and f_* both preserve fibred finite limits, we get induced left exact functors $\overline{f}^*$ and $\overline{f}_*$ between $\mathbb{B}[\mathsf{p}]$ and $\mathbb{B}[\mathsf{q}]$ defined as above. In fact, since f^* preserves existential quantification (as a fibred left adjoint), $\overline{f}^*$ may be constructed without recourse to completions.

Proposition 5.4.6. *Let* $f = (f^*, f_*)$ *be a geometric morphism of triposes, as in*

$$\begin{tikzcd}
\mathbb{P} \arrow[rr, bend left, "f_*"] \arrow[rr, phantom, "\top"] \arrow[dr, "\mathsf{p}"'] & & \mathbb{Q} \arrow[ll, bend left, "f^*"] \arrow[dl, "\mathsf{q}"] \\
& \mathbb{B}. &
\end{tikzcd}$$

Then $\overline{f} = (\overline{f}^*, \overline{f}_*)\colon \mathbb{B}[\mathsf{p}] \to \mathbb{B}[\mathsf{q}]$ *is a geometric morphism of toposes.*

Proof Sketch. Suppose, without loss of generality by Proposition 5.4.3, that $(X, \approx) \in \mathbb{B}[\mathsf{p}]$ is complete. The counit $\epsilon_{(X,\approx)}\colon \overline{f}^*\overline{f}_*(X,\approx) \to (X,\approx)$ is represented by the functional relation

$$E(x,x') = f^*f_*(x \approx x) \wedge x \approx x'.$$

For any $g = [G]\colon \overline{f}^*(Y,\approx) \to (X,\approx)$, the associated unique morphism $\overline{g}\colon (Y,\approx) \to \overline{f}_*(X,\approx)$, as in the diagram

$$\begin{tikzcd} \overline{f}_*(X,\approx_X) \\ (Y,\approx_Y) \arrow[u, dashed, "\overline{g}"] \end{tikzcd} \qquad \begin{tikzcd} \overline{f}^*\overline{f}_*(X,\approx_X) \arrow[r, "\epsilon"] & (X,\approx_X) \\ \overline{f}^*(Y,\approx_Y) \arrow[u, "\overline{f}^*\overline{g}"] \arrow[ur, "g"'] \end{tikzcd}$$

is represented by the functional relation $\overline{G}$ given by

$$\overline{G}(y,x) = f_*(G(y,x)) \wedge (y \approx y).$$

See [Pit81] for more details. □

The following observation is easy, but useful in the following. It has probably been know to Hyland and Pitts, but I have not seen it written down anywhere, so I include a proof here.

Proposition 5.4.7. *Let* $\mathbb{B}$ *be a finitely complete category and let* $\begin{array}{c}\mathbb{P}\\ \downarrow\mathsf{p}\\ \mathbb{B}\end{array}$ *and* $\begin{array}{c}\mathbb{Q}\\ \downarrow\mathsf{q}\\ \mathbb{B}\end{array}$ *be* $\mathbb{B}$*-triposes. Suppose* $f = (f^*, f_*)\colon \mathsf{p} \to \mathsf{q}$ *is a geometric morphism of triposes. Suppose further that* f^* *is full and faithful and that* f_* *preserves existential quantification. Then also the induced functor* $\overline{f}^*$ *is full and and faithful.*

Note that whenever f_* has a fibred right adjoint, it preserves existential quantification (as a fibred left adjoint).

Proof. We show that the unit $\eta\colon id \Rightarrow \overline{f}_*\overline{f}^*$ is an isomorphism, from which it follows that $\overline{f}^*$ is full and faithful [Mac71, Dual of Theorem 1, Section IV.3].

Now since f_* preserves existential quantification, both $\overline{f}_*$ and $\overline{f}^*$ may be constructed without recourse to completions so that $\overline{f}^*(X,\approx) = (X, f^* \approx)$ and $\overline{f}_*(Y,\approx) = (Y, f_* \approx)$. The unit $\eta\colon id \Rightarrow \overline{f}_*\overline{f}^*$ is then given by the functional relation $\eta(x,x') = |x \approx x \wedge f_*f^*(x \approx x')|\colon (X,\approx) \to (X, f_*f^* \approx)$. But $f_*f^* \cong id$ by the assumption that f^* is full and faithful, so $\eta(x,x') \cong x \approx x'$, the identity on $(X,\approx)$, as required. □

The following theorem was known to Martin Hyland but apparently has never been published.[2] We include a proof here. See [Joh77, Joh79a, Joh81] for more on localic geometric morphisms.

Theorem 5.4.8. *Let $\mathbb{B}$ be a finitely complete category and let* $\begin{smallmatrix}\mathbb{P}\\ \downarrow \mathsf{p}\\ \mathbb{B}\end{smallmatrix}$ *and* $\begin{smallmatrix}\mathbb{Q}\\ \downarrow \mathsf{q}\\ \mathbb{B}\end{smallmatrix}$ *be $\mathbb{B}$-triposes. Suppose $f = (f^*, f_*)\colon \mathsf{p} \to \mathsf{q}$ is a geometric morphism of triposes. Then $\mathbb{B}[\mathsf{p}]$ is localic over $\mathbb{B}[\mathsf{q}]$ via the induced geometric morphism $\overline{f} = (\overline{f}^*, \overline{f}_*)\colon \mathbb{B}[\mathsf{p}] \to \mathbb{B}[\mathsf{q}]$.*

Proof. We want to prove that $\mathbb{B}[\mathsf{p}]$ is equivalent to the category of $\mathbb{B}[\mathsf{q}]$-valued sheaves on the internal locale $\overline{f}_*(\Omega_{\mathbb{B}[\mathsf{p}]})$ in $\mathbb{B}[\mathsf{q}]$. As usual [Joh77, Joh79a, Joh81] it suffices to show that, for all $X \in \mathbb{B}[\mathsf{p}]$, there exists a $Y \in \mathbb{B}[\mathsf{q}]$ and a diagram

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & \overline{f}^* Y \\ X & \end{tikzcd}$$

in $\mathbb{B}[\mathsf{p}]$ presenting X as a subquotient of $\overline{f}^*Y$ for Y an object of $\mathbb{B}[\mathsf{q}]$. Write $\nabla_{\mathsf{p}}\colon \mathbb{B} \to \mathbb{B}[\mathsf{p}]$ for the constant objects functor for p. Then, as noted in Section 5.3, for any $X \in \mathbb{B}[\mathsf{p}]$, there exists an object $I \in \mathbb{B}$ and a diagram

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & \nabla_{\mathsf{p}}(I) = (I, \exists_{\delta_I}(T)) \\ X & \end{tikzcd} \tag{5.1}$$

in $\mathbb{B}[\mathsf{p}]$ presenting X as a subquotient of a constant object $\nabla_{\mathsf{p}}(I)$. Now since f^* is the inverse image of a geometric morphism of triposes, f^* preserves existential quantification (as a fibred left adjoint), so $\overline{f}^*(\nabla_{\mathsf{q}}(I)) \cong \nabla_{\mathsf{p}}(I)$, and the diagram in (5.1) is the required diagram. □

5.5 Topologies and Sub-Triposes

The notion of a Lawvere-Tierney topology on a tripos is a generalization of the notion of a nucleus (or j-operator) on a locale and just as a nucleus

[2] Martin Hyland suggested me to prove an instance of this result for some specific realizability triposes. I found a proof and saw that it applied in general to arbitrary triposes as shown here; a fact that Hyland, as expected, was aware of.

on a locale gives rise to a subtopos of the topos of sheaves on the locale, a Lawvere-Tierney topology on a tripos gives rise to a subtopos of the topos resulting from the tripos.

We confine ourselves to the case when the tripos p is a canonically presented tripos on an object Σ in a topos $\mathcal{E}$, *i.e.*, when the fibre over I is $\mathcal{E}(I,\Sigma)$, and reindexing is given by composition.

Definition 5.5.1. A **(Lawvere-Tierney) topology** on a canonically presented $\mathcal{E}$-tripos $\begin{smallmatrix}\mathbb{P}\\ \downarrow \mathsf{p}\\ \mathcal{E}\end{smallmatrix}$ is an inflationary, idempotent, left exact fibred functor $J\colon \mathbb{P} \to \mathbb{P}$ over $\mathcal{E}$.

Such a topology on $\mathsf{p} = \mathcal{E}(-,\Sigma)$ can be specified by a map $J\colon \Sigma \to \Sigma$ in $\mathcal{E}$ satisfying

- $p, q\colon \Sigma \mid \emptyset \vdash (p \supset q) \supset (Jp \supset Jq)$
- $\emptyset \mid \emptyset \vdash J(\top)$
- $p\colon \Sigma \mid \emptyset \vdash J(Jp) \supset Jp$

in the logic of p.

Such a map $J\colon \Sigma \to \Sigma$ represents a strict predicate on $\Omega = (\Sigma, \supset\!\subset) \in \mathcal{E}[\mathsf{p}]$, which is the generic j-dense subobject of Ω. That is, the classifying map of the subobject $\|J\| \rightarrowtail \Omega$ is a Lawvere-Tierney topology $j\colon \Omega \to \Omega$ in $\mathcal{E}[\mathsf{p}]$, as in

$$\begin{tikzcd} \|J\| \arrow[r] \arrow[d, tail] & 1 \arrow[d, "\top"] \\ \Omega \arrow[r, "j"'] & \Omega, \end{tikzcd}$$

where, recall, $\|J\| = (\Sigma, \approx_J)$ with $|p \approx_J q| = (p \supset\!\subset q) \wedge J(p)$.

Let $\mathrm{Sh}_j\mathcal{E}[\mathsf{p}]$ be the sheaf subtopos corresponding to a topology J on p. Then $\mathrm{Sh}_j\mathcal{E}[\mathsf{p}]$ is equivalent to $\mathcal{E}[\mathsf{p}_J]$ for some canonically presented $\mathcal{E}$-tripos p_J, described as follows. Tripos p_J is also canonically presented on Σ, but $\supset$, $\forall$, and D are redefined by letting

- $\supset_{\mathsf{p}_J}$ be $\Sigma \times \Sigma \xrightarrow{id \times J} \Sigma \times \Sigma \xrightarrow{\supset_P} \Sigma$, *i.e.*, $(\varphi \supset_{\mathsf{p}_J} \psi) = (\varphi \supset_P J\psi)$,
- $(\forall_F)(\varphi)$ be $(\forall_F)(J\varphi)$,
- $1 \xrightarrow{p} \Sigma$ be in $\mathsf{D}_{\mathsf{p}_J}$ iff $1 \xrightarrow{p} \Sigma \xrightarrow{J} \Sigma$ is in D_{p}.

Thus $\varphi \vdash^{\mathsf{p}_J} \psi$ iff $\varphi \vdash^{\mathsf{p}} J\psi$, whilst $\top$, $\wedge$, $\bot$, $\vee$, and $\exists$ remain unchanged.

Let $f = (f^*, f_*) \colon \mathsf{p} \to \mathsf{q}$ be a geometric morphisms between $\mathcal{E}$-triposes. Then $J = f_* f^*$ is a topology on q and the surjection-inclusion factorization [MM92] of the induced geometric morphism $\overline{f} \colon \mathcal{E}[\mathsf{p}] \to \mathcal{E}[\mathsf{q}]$ takes the form

$$\mathcal{E}[\mathsf{p}] \longrightarrow \mathcal{E}[\mathsf{q}_J] \hookrightarrow \mathcal{E}[\mathsf{q}].$$

Thus $\overline{f}$ is an inclusion iff $f^* f_* \cong id$, and in this case we shall say that f is an **inclusion of triposes**.

Chapter 6

The Relative Realizability Topos $\mathrm{RT}(A, A_\sharp)$

In the present chapter we initiate our study of the relative realizability topos $\mathrm{RT}(A, A_\sharp)$ obtained from the relative realizability tripos in Section 5.1.4 by the standard tripos-to-topos construction. We show how $\mathrm{RT}(A, A_\sharp)$ relates the standard realizability toposes $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A)$; in particular, we prove that there is a localic local geometric morphism from $\mathrm{RT}(A, A_\sharp)$ to $\mathrm{RT}(A_\sharp)$. In Chapters 7–9 we study local geometric morphisms at an abstract level, and in Chapter 10 we then return to study $\mathrm{RT}(A, A_\sharp)$ in more detail.

For the remainder of this chapter, we let A be a PCA and let $A_\sharp \subseteq A$ be a sub-PCA of A. Recall from the introduction in Chapter 1 that we are thinking of the realizers from A as "continuous" and of the realizers from $A_\sharp$ as "computable." Thus we shall call elements of A **continuous realizers** and elements of $A_\sharp$ **computable realizers**.

Example 6.0.2. There are many examples of PCA's A with a sub-PCA $A_\sharp$. Here are a few:

1. $A = \mathbb{P}$, the graph model of the lambda calculus, see Example 3.1.15, and let $A_\sharp = RE$, the recursively enumerable graph model, see Example 3.1.16. Note that $\mathbb{P}$ has a continuum of (countable) sub-PCA's.

2. A is Kleene's second model N^N, see [Bee85, Section VI.7.4], and $A_\sharp$ is the sub-model of recursive functions from N to N, see [Bee85, Section VI.7.5].

3. A is van Oosten's combinatory algebra $\mathcal{B}$ for sequential computation and $A_\sharp$ is its effective subalgebra $\mathcal{B}_{\mathit{eff}}$, see [vO99, Lon98].

Convention 6.0.3. Let $\begin{array}{c}\mathbf{UFam}(A)\\ \downarrow\mathsf{p}\\ \mathbf{Set}\end{array}$ and $\begin{array}{c}\mathbf{UFam}(A_\sharp)\\ \downarrow\mathsf{q}\\ \mathbf{Set}\end{array}$ be the standard realizability tripos over A and $A_\sharp$, respectively, and let $\begin{array}{c}\mathbf{UFam}(A,A_\sharp)\\ \downarrow\mathsf{r}\\ \mathbf{Set}\end{array}$ be the relative realizability tripos over A and $A_\sharp$. We denote the resulting realizability toposes in the following way:

$$\begin{aligned}\mathrm{RT}(A) &= \mathbf{Set}[\mathsf{p}]\\ \mathrm{RT}(A_\sharp) &= \mathbf{Set}[\mathsf{q}]\\ \mathrm{RT}(A, A_\sharp) &= \mathbf{Set}[\mathsf{r}].\end{aligned}$$

Before going into the relationship between $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A)$ and $\mathrm{RT}(A_\sharp)$, let us for emphasis write out explicitly what the object and morphisms of $\mathrm{RT}(A, A_\sharp)$ are. Objects of $\mathrm{RT}(A, A_\sharp)$ are pairs $(X, \approx_X)$ with $X \in \mathbf{Set}$ a set and $\approx_X : X \times X \to PA$ a non-standard equality predicate with computable realizers for symmetry and transitivity. Thus we require that both

$$\Big(A_\sharp \cap \bigcap_{x,x' \in X} \big((x \approx_X x') \supset (x' \approx_X x)\big)\Big) \neq \emptyset$$

$$\Big(A_\sharp \cap \bigcap_{x,x',x'' \in X} \big((x \approx_X x') \wedge (x' \approx_X x'') \supset (x \approx_X x'')\big)\Big) \neq \emptyset,$$

hold, where, for $p, q \in PA$,

$$\begin{aligned}p \wedge q &= \{\, \langle a, b\rangle \mid a \in p \text{ and } b \in q \,\}\\ p \supset q &= \{\, a \in A \mid \forall b \in p.\ a \cdot b \in q \,\},\end{aligned}$$

see Section 5.1.4.

A morphism $f : (X, \approx_X) \to (Y, \approx_Y)$ is an equivalence class $f = [F]$ of (PA)-valued functional relations $F : X \times Y \to PA$ with computable realizers for functionality. That is, we require that

$$\Big(A_\sharp \cap \bigcap_{x,x' \in X, y,y' \in Y} \big((x \approx_X x') \wedge (y \approx_Y y') \wedge F(x,y) \supset F(x',y')\big)\Big) \neq \emptyset$$

$$\Big(A_\sharp \cap \bigcap_{x \in X, yY} \big(F(x,y) \supset (x \approx_X x) \wedge (y \approx_Y y)\big)\Big) \neq \emptyset$$

$$\Big(A_\sharp \cap \bigcap_{x, \in X, y,y' \in Y} \big(F(x,y) \wedge F(x,y') \supset (y \approx_Y y')\big)\Big) \neq \emptyset$$

$$\Big(A_\sharp \cap \bigcap_{x, \in X} \big((x \approx_X x) \supset \bigcup_{y \in Y} F(x,y)\big)\Big) \neq \emptyset$$

all hold.

Two such F and F' are equivalent just in case

$$\Big(A_\sharp \cap \bigcap_{x,\in X, y\in Y} \big(F(x,y) \supset\!\subset F'(x,y)\big)\Big) \neq \emptyset.$$

We now see that, intuitively speaking, it makes sense to think of objects of $\mathrm{RT}(A, A_\sharp)$ as objects with *continuous* realizers for existence and equality of objects, and of morphisms $f = [F]$ as *computable* maps, since the realizers for functionality of F are required to be computable. Thus the slogan is:

Slogan. $\mathrm{RT}(A, A_\sharp)$ *has "continuous objects and computable morphisms."*

Remark 6.0.4. The relative realizability tripos $\begin{smallmatrix}\mathbf{UFam}(A,A_\sharp)\\ \downarrow \mathsf{r}\\ \mathbf{Set}\end{smallmatrix}$ underlying the topos $\mathrm{RT}(A, A_\sharp)$ can also be obtained from a WCPC-category: consider the monoid $\mathrm{M}(A, A_\sharp)$ of $A_\sharp$-definable partial functions from A to A and the functor $U_0 \colon \mathrm{M}(A, A_\sharp) \to \mathbf{Ptl}$ the inclusion functor. Let $\mathbb{C} = \mathrm{Split}(\mathrm{M}(A, A_\sharp), U_0)$ and let $U = \mathrm{Split}(U_0)$ (see Page 62). Then $\mathbb{C}$ is a WCPC-category and the pretripos generated by $\mathbb{C}$ and U is equivalent to the realizability tripos $\begin{smallmatrix}\mathbf{UFam}(A,A_\sharp)\\ \downarrow \mathsf{r}\\ \mathbf{Set}\end{smallmatrix}$.

6.1 On the Relation Between $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A)$

Recall from Section 5.1.4 that the inhabited subsets of A form a filter Φ in the fibre over 1 in the tripos r and that tripos p is exactly r_Φ, see Remark 5.1.3. Then as in [Pit81, Page 26], we can define a filter $\hat{\Phi}$ of subobjects of 1 in $\mathrm{RT}(A, A_\sharp) = \mathbf{Set}[\mathsf{r}]$ consisting of

$$\|\varphi\| \rightarrowtail 1,$$

with $\|\varphi\| = (1, (*, *) \mapsto \varphi(*))$, for each $\varphi \in \Phi$. Then as Pitts remarks [Pit81, Page 26], the filter-quotient $\mathrm{RT}(A, A_\sharp)_{\hat{\Phi}}$ is equivalent to $\mathbf{Set}[\mathsf{r}_\Phi] = \mathbf{Set}[\mathsf{p}] = \mathrm{RT}(A)$. Under this equivalence, the logical functor $\mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A, A_\sharp)_{\hat{\Phi}}$ is identified with the obvious functor from $\mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A)$ which is the identity on objects. See [MM92, Joh77, LM82] for more on filter-quotients (also called filter-powers).

6.2 On the Relation Between $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A_\sharp)$

We now define three fibred functors over **Set** among the triposes q and r underlying $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A_\sharp)$, as in

$$\begin{tikzcd}
\mathbf{UFam}(A_\sharp) \arrow[rr, bend left, "\Delta"] \arrow[rr, bend right, "\nabla"'] \arrow[dr, "\mathsf{q}"'] & & \mathbf{UFam}(A, A_\sharp) \arrow[ll, "\Gamma"'] \arrow[dl, "\mathsf{r}"] \\
& \mathbf{Set}. &
\end{tikzcd}$$

The functors are defined by

$$\begin{aligned}
\Delta(\psi\colon I \to PA_\sharp) &= \psi \\
\Gamma(\varphi\colon I \to PA) &= \lambda i.\ A_\sharp \cap \varphi(i) \\
\nabla(\psi\colon I \to PA_\sharp) &= \lambda i.\ \bigcup_{\varphi \in PA} \Big(\varphi \wedge \big(A_\sharp \cap \varphi \supset \psi(i)\big)\Big),
\end{aligned}$$

with $\wedge$ and $\supset$ calculated as in r's fibre over 1, *i.e.*, in $\mathbf{UFam}(A, A_\sharp)_1$.

The above equations give the action of the functors on objects. The action of each of the functors on a morphism is the identity action. (Recall that in $\mathbf{UFam}(A_\sharp)$ there is a morphism $u\colon \psi \to \psi'$, with $\psi \in \mathbf{UFam}(A_\sharp)_I$ and $\psi' \in \mathbf{UFam}(A_\sharp)_J$ exactly if ψ is less than $\psi' \circ u$ in the fibre $\mathbf{UFam}(A_\sharp)_I$. Likewise for $\mathbf{UFam}(A, A_\sharp)$.) It is easy to see that Δ and Γ are well-defined fibred functors and that ∇ preserves cartesian morphisms (recall that the cartesian morphism over u is u itself) and that $\mathsf{r}\nabla = \mathsf{q}$. To verify that the functors are well-defined, it thus only remains to verify the functoriality of ∇. To this end, suppose that $u : \psi \to \psi'$ in $\mathbf{UFam}(A_\sharp)$, *i.e.*, that $\psi \leq_{\mathbf{UFam}(A_\sharp)_I} \psi' \circ u$. Then there is a realizer $c \in A_\sharp$ such that

$$c \in \bigcap_{i \in I} \big(\psi(i) \supset \psi'(u(i))\big).$$

Let

$$d = \lambda x.\ \langle \pi x, \lambda y.\ c(\pi'(x)(y))\rangle.$$

Then $d \in A_\sharp$ (since $c \in A_\sharp$) and it is easy to verify that

$$d \in \bigcap_{i \in I} \big(\nabla(\psi)(i) \supset \nabla(\psi')(u(i)\big),$$

as required.

Theorem 6.2.1. *Under these definitions it follows that*

1. (Δ, Γ) *is a geometric morphism of triposes from* r *to* q.
2. (Γ, ∇) *is a geometric morphism of triposes from* q *to* r.
3. *For all* $I \in \mathbf{Set}$, Δ_I *and* ∇_I *are both full and faithful.*

Remark 6.2.2. Item 3 is equivalent to Δ and ∇ being full and faithful [Jac99, Exercise 1.7.2].

Proof. It is easy to see that Δ is left adjoint to Γ using that $A_\sharp$ is closed under the partial application of A. Further, it is clear that Δ preserves finite limits and is full and faithful since $\vdash^{\mathsf{q}}_I$ and $\vdash^{\mathsf{r}}_I$ are defined in the same way (requiring computable realizers).

By Lemma 2.2.11 it suffices to show that ∇_I is left adjoint to Γ_I, for all $I \in \mathbf{Set}$ (the Beck-Chevalley condition in Lemma 2.2.11 is easily seen to hold). Since q and r are both fibred preorders, we just have to show that

$$\varphi \vdash^{\mathsf{r}}_I \nabla\psi \iff \Gamma\varphi \vdash^{\mathsf{q}}_I \psi,$$

for all $\varphi \in \mathbf{UFam}(A, A_\sharp)_I$ and all $\psi \in \mathbf{UFam}(A_\sharp)_I$. To this end, suppose $\varphi \vdash^{\mathsf{r}}_I \nabla\psi$, via a realizer $c \in A_\sharp$. Let

$$d = \lambda x.\ \pi(c(x))(\pi'(c(x))) \in A_\sharp.$$

It is easy to verify that d is a a realizer for $\Gamma\varphi \vdash^{\mathsf{q}}_I \psi$.

For the other direction, suppose $d \in A_\sharp$ is a realizer for $\Gamma\varphi \vdash^{\mathsf{q}}_I \psi$. Then

$$c = \lambda x.\ \langle x, \lambda y.\ d(y)\rangle \in A_\sharp$$

is a realizer for $\varphi \vdash^{\mathsf{r}}_I \nabla\psi$.

Since Δ is full and faithful and since $\Delta \dashv \Gamma \dashv \nabla$, also ∇ is full and faithful [MM92, Lemma 1, Section VII.4], completing the proof of the theorem. □

By Proposition 5.4.6 these geometric morphisms (Δ, Γ) and (Γ, ∇) of triposes lift to two geometric morphisms between the induced toposes, as in

$$\begin{tikzcd} \mathrm{RT}(A_\sharp) \arrow[r, bend left, "\Delta"] \arrow[r, bend right, "\nabla"'] & \mathrm{RT}(A, A_\sharp) \arrow[l, "\Gamma"'] \end{tikzcd}, \qquad \Delta \dashv \Gamma \dashv \nabla.$$

(Here we do not distinguish notationally between the functors at the tripos level and at the topos level). In particular, Δ preserves finite limits. By

Proposition 5.4.7, $\Delta\colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ is full and faithful and thus, by [MM92, Lemma 1, Section VII.4], also $\nabla\colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ is full and faithful. Hence $\Gamma\Delta \cong 1 \cong \Gamma\nabla$. The geometric morphisms (Δ, Γ) is therefore a connected surjective geometric morphism and $(\Gamma, \nabla)\colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ is an embedding (see [MM92, Chapter VII] or [Joh77, Chapter 4] for more on different classes of geometric morphisms). By Theorem 5.4.8, $\mathrm{RT}(A, A_\sharp)$ is localic over $\mathrm{RT}(A_\sharp)$ via the geometric morphism (Δ, Γ). Summarizing we have:

Theorem 6.2.3. *The geometric morphism* $(\Delta, \Gamma) : \mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A_\sharp)$ *is a* localic local map of toposes.

Proof. (Δ, Γ) is localic as remarked above and it is local since Γ has a right adjoint ∇ for which $\Gamma\nabla \cong 1$. ☐

Local maps of toposes have been studied by Johnstone and Moerdijk [JM89] and provide an instance of what Lawvere has called *unity and identity of opposites* [Law91, Law89]. We will have a lot more to say about local maps in the following chapters.

For future use we now state explicitly some of the data given by the adjunction $\Delta \dashv \Gamma$.

Consider first the adjunction $\Delta \dashv \Gamma$. Since Γ at the level of triposes has a right adjoint, both Δ and Γ are defined defined without recourse to completions so that $\Delta(Y, \approx_Y) = (Y, \approx_Y)$ and $\Gamma(X, \approx_X) = (X, \Gamma \approx_X)$. As already observed, the unit $\eta\colon 1 \Rightarrow \Gamma\Delta$ is naturally isomorphic to the identity. The counit $\epsilon\colon \Delta\Gamma \Rightarrow 1$ at an object $(X, \approx) \in \mathrm{RT}(A, A_\sharp)$ is represented by the functional relation E given by

$$\begin{aligned} E(x, x') &= \Delta\Gamma(x \approx x) \wedge (x \approx x') \\ &= (x \approx x) \cap A_\sharp \wedge (x \approx x') \end{aligned}$$

(see the proof sketch of Proposition 5.4.6). For a morphism

$$g = [G]\colon \Delta(Y, \approx_Y) \to (X, \approx_X)$$

in $\mathrm{RT}(A, A_\sharp)$, the associated unique morphism $\overline{g}\colon (Y, \approx_Y) \to \Gamma(X, \approx_X)$, as in the diagram

$$\begin{tikzcd} \Gamma(X, \approx_X) & & \Delta\Gamma(X, \approx_X) \arrow[r, "\epsilon"] & (X, \approx_X) \\ (Y, \approx_Y) \arrow[u, dashed, "\overline{g}"] & & \Delta(Y, \approx_Y) \arrow[u, "\Delta\overline{g}"] \arrow[ur, "g"'] & \end{tikzcd}$$

is represented by the functional relation $\overline{G}$ given by

$$\begin{aligned}\overline{G}(y,x) &= \Gamma(G(y,x)) \wedge (y \approx_Y y) \\ &= G(y,x) \cap A_\sharp \wedge (y \approx_Y y) \\ &\cong G(y,x) \cap A_\sharp\end{aligned}$$

(the last isomorphism since G is strict and Δ is the identity).

Regarding the adjunction $\Gamma \dashv \nabla$, note that since ∇ at the level of triposes does not preserve existentials, $\nabla\colon\ \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ is constructed using completions and is therefore not so easily described explicitly. Since we shall not need to calculate with this adjunction explicitly, we do not include a detailed treatment of it here.

Geometric Morphisms from Set to $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A_\sharp)$

Following Pitts [Pit81, Examples 4.9, Page 53] there is a geometric morphism of triposes as in

$$\begin{tikzcd} \mathrm{Sub}(\mathbf{Set}) \arrow[rr, shift left, "\delta_{\mathsf{q}}"] \arrow[rr, phantom, "\top"] \arrow[dr] & & \mathbf{UFam}(A_\sharp) \arrow[ll, shift left, "\gamma_{\mathsf{q}}"] \arrow[dl, "\mathsf{q}"] \\ & \mathbf{Set} & \end{tikzcd}$$

explicitly given by

$$\gamma_{\mathsf{q}}(\psi\colon I \to PA_\sharp) = \{\, i \in I \mid \psi(i) \neq \emptyset \,\}$$

$$\delta_{\mathsf{q}}(I' \subseteq I) = i \mapsto \begin{cases} A_\sharp & \text{if } i \in I', \\ \emptyset & \text{otherwise.} \end{cases}$$

We denote the resulting geometric morphism by $(\Gamma_{\mathsf{q}}, \nabla_{\mathsf{q}}) = (\overline{\gamma_{\mathsf{q}}}, \overline{\delta_{\mathsf{q}}})$, as in

$$\begin{tikzcd} \mathbf{Set} \arrow[r, shift left, "\nabla_q"] \arrow[r, phantom, "\top"] & \mathrm{RT}(A_\sharp) \arrow[l, shift left, "\Gamma_q"] \end{tikzcd}$$

As the notation suggests, ∇_{q} is indeed the constant objects functor from Section 5.3.

Explicitly, Γ_{q} may be described as follows. An object $(I, \approx_I)$ is mapped to the quotient set $\mathrm{Dom}(\sim)/\!\sim$, where $i \sim i \iff |i \approx_I i| \neq \emptyset$. A morphism $f = [F] : (I, \approx) \to (J, \approx)$ is mapped to the function which maps $[i]$ to $[j]$

iff $F(i,j) \neq \emptyset$. The functor Γ_{q} is isomorphic to the global sections functor $\mathrm{Hom}_{\mathrm{RT}(A_\sharp)}(1, -)$.

Explicitly, the functor ∇_{q} maps an object $I \in \mathbf{Set}$ to the object $(I, \approx_I)$ with

$$|i \approx_I i| = \begin{cases} A_\sharp & \text{if } i = i', \\ \emptyset & \text{otherwise.} \end{cases}$$

A function $f : I \to J$ is mapped to the morphism $[F]$ with

$$F(i,j) = \begin{cases} A_\sharp & \text{if } f(i) = j \\ \emptyset & \text{otherwise.} \end{cases}$$

In much the same way, we get an geometric morphism of triposes as in

$$\begin{tikzcd} \mathrm{Sub}(\mathbf{Set}) \arrow[rr, shift left, "\delta_{\mathsf{r}}"] \arrow[rr, phantom, "\top"] \arrow[dr] & & \mathbf{UFam}(A, A_\sharp) \arrow[ll, shift left, "\gamma_{\mathsf{r}}"] \arrow[dl, "\mathsf{r}"] \\ & \mathbf{Set} & \end{tikzcd}$$

with the functors explicitly given by

$$\gamma_{\mathsf{r}}(\varphi \colon I \to PA) = \{\, i \in I \mid \varphi(i) \neq \emptyset \,\}$$
$$\delta_p(I' \subseteq I) = i \mapsto \begin{cases} A & \text{if } i \in I', \\ \emptyset & \text{otherwise.} \end{cases}$$

We denote the resulting geometric morphism by $(\Gamma_{\mathsf{r}}, \nabla_{\mathsf{r}}) = (\overline{\gamma_{\mathsf{r}}}, \overline{\delta_{\mathsf{r}}})$, as in

$$\begin{tikzcd} \mathbf{Set} \arrow[r, shift left, "\nabla_{\mathsf{r}}"] \arrow[r, phantom, "\top"] & \mathrm{RT}(A, A_\sharp). \arrow[l, shift left, "\Gamma_{\mathsf{r}}"] \end{tikzcd}$$

Explicitly, Γ_{r} may be described analogously to Γ_{q}, *i.e.*, an object $(I, \approx_I)$ is mapped to quotient set $\mathrm{Dom}(\sim)/{\sim}$, where $i \sim i \iff |i \approx_I i| \neq \emptyset$. But notice that Γ_{r} is *not* (isomorphic to) the global sections functor $\mathrm{Hom}_{\mathrm{RT}(A,A_\sharp)}(1, -)$! For a concrete counter-example, suppose $A_\sharp$ is a proper sub-PCA of A and consider the object $(I, \approx_I)$ in $\mathrm{RT}(A, A_\sharp)$ with $|i \approx_I i'| = A \setminus A_\sharp$. Then $\Gamma_{\mathsf{r}}(I, \approx_I)$ is isomorphic to the terminal object of $\mathbf{Set}$, a one-element set. But $\mathrm{Hom}_{\mathrm{RT}(A,A_\sharp)}(1, (I, \approx_I))$ is empty (there is no functional relation with a realizer *in* $A_\sharp$ for totality).

The functor ∇_{r} maps an object $I \in \mathbf{Set}$ to the object $(I, \approx_I)$ with

$$|i \approx_I i| \stackrel{\text{def}}{=} \begin{cases} A & \text{if } i = i', \\ \emptyset & \text{otherwise.} \end{cases}$$

A function $f : I \to J$ is mapped by ∇_{r} to the morphism $[F]$ with

$$F(i, j) = \begin{cases} A & \text{if } f(i) = j, \\ \emptyset & \text{otherwise.} \end{cases}$$

For the record state the following observation.

Proposition 6.2.4. *The functors δ_{q} and δ_{r} preserve coproducts (existential quantification $\exists$).*

Proof. We just consider δ_{q}; the reasoning is similar for δ_{r}. Let $u : I \to J$ in **Set**. For $I' \subseteq I$, let $u_*(I')$ denote the image of I' under u. Then

$$\begin{aligned} \delta_{\mathsf{q}}(\exists_u(I' \subseteq I)) &= \delta_{\mathsf{q}}(u_*(I') \subseteq J) \\ &= j \mapsto \begin{cases} A_\sharp & \text{if } j \in u_*(I'), \\ \emptyset & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \exists_u(\delta_{\mathsf{q}}(I' \subseteq I)) &= \exists_u(\psi) \quad \text{where } \psi(i) = A_\sharp \text{ if } i \in I',\ \emptyset \text{ otherwise} \\ &= j \mapsto \bigcup \{\, \psi(i) \mid u(i) = j \,\} \\ &= j \mapsto \begin{cases} A_\sharp & \text{if } \exists i \in I'.u(i) = j \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Since $\delta_{\mathsf{q}}(\exists_u(I' \subseteq I)) = \exists_u(\delta_{\mathsf{q}}(I' \subseteq I))$, we have the required. $\square$

Theorem 6.2.5. *Consider the following diagram, all fibred over* **Set***,*

$$\begin{tikzcd}
\mathbf{UFam}(A_\sharp) \arrow[rr, shift left=3, "\Delta"] \arrow[rr, shift right=3, "\nabla"'] \arrow[dr, bend right, "\gamma_{\mathsf{q}}"'] & & \mathbf{UFam}(A, A_\sharp) \arrow[ll, "\Gamma"'] \arrow[dl, bend left, "\gamma_{\mathsf{r}}"] \\
& \mathrm{Sub}(\mathbf{Set}) \arrow[ul, "\delta_{\mathsf{q}}"] \arrow[ur, "\delta_{\mathsf{r}}"'] &
\end{tikzcd}
\qquad
\begin{aligned} &\Delta \dashv \Gamma \dashv \nabla, \\ &\gamma_{\mathsf{q}} \dashv \delta_{\mathsf{q}} \\ &\gamma_{\mathsf{r}} \dashv \delta_{\mathsf{r}}. \end{aligned}$$

Then

1. $\Delta \circ \delta_q \cong \delta_p$
2. $\Gamma \circ \delta_p \cong \delta_q$
3. $\gamma_q \circ \Gamma \dashv \nabla \circ \delta_q$

Proof. Straightforward. □

This theorem then extends to the level of toposes in the obvious way. Explicitly we have the following.

Theorem 6.2.6. *Consider the following diagram of toposes and functors:*

$$\begin{tikzcd}
\mathrm{RT}(A_\sharp) \arrow[rr, bend left, "\Delta"] \arrow[rr, bend right, "\nabla"'] \arrow[dr, shift right, "\Gamma_{\mathsf{q}}"'] & & \mathrm{RT}(A, A_\sharp) \arrow[ll, "\Gamma"] \arrow[dl, shift left, "\Gamma_{\mathsf{r}}"] \\
& \mathbf{Set} \arrow[ul, shift right, "\nabla_{\mathsf{q}}"'] \arrow[ur, shift left, "\nabla_{\mathsf{r}}"] &
\end{tikzcd}
\qquad
\begin{array}{l}
\Delta \dashv \Gamma \dashv \nabla, \\
\Gamma_{\mathsf{q}} \dashv \nabla_{\mathsf{q}} \\
\Gamma_{\mathsf{r}} \dashv \nabla_{\mathsf{r}}.
\end{array}$$

Then

1. $\Delta \circ \nabla_{\mathsf{q}} \cong \nabla_{\mathsf{r}}$
2. $\Gamma \circ \nabla_{\mathsf{r}} \cong \nabla_{\mathsf{q}}$
3. $\Gamma_{\mathsf{q}} \circ \Gamma \dashv \nabla \circ \Delta_{\mathsf{r}}$

Chapter 7

An Elementary Axiomatization of Local Maps of Toposes

In this chapter we present an elementary axiomatization of local maps of toposes. The axioms are shown to be sound and complete in the sense that whenever a topos satisfies the axioms then it gives rise to a local map and, moreover, any local map of toposes satisfies the axioms. Below we first recall the definition of a local map of toposes in Section 7.1. In Section 7.2 we then describe the approach to the axiomatization that we will take and we recall some material on orthogonal and coorthogonal categories from [KL89]. Finally, in Section 7.3 we present our axiomatization of local maps and show that they are sound and complete. We first present a number of definitions of relevant concepts and prove some properties of these concepts before we suggest the actual axioms and prove that they are sound and complete. Our development will be done in category-theoretic language but sometimes we also include corresponding treatments phrased in the internal language of the relevant toposes.

As explained in the previous chapter the axiomatization presented here is motivated by our study of the relationship between the relative realizability topos $\mathrm{RT}(A, A_\sharp)$ and the realizability topos $\mathrm{RT}(A_\sharp)$. In this chapter, however, we stay completely general and concern ourselves with *arbitrary* local maps of toposes. It can thus be read independently of the rest of the thesis.

7.1 Local Maps of Toposes

Before recalling the definition of a local map, we recall the standard notion of a bounded geometric morphism [Joh77, Section 4.4]. For explanatory remarks on this definition, see *loc. cit.*. All of this chapter, except Section 7.3.2, and also the following chapters, in which we focus on localic local maps, can be read without worrying about boundedness.

Definition 7.1.1. Let $f = (f^*, f_*) \colon \mathcal{E} \to \mathcal{F}$ be a geometric morphism and let G be an object of $\mathcal{E}$.

1. G is an **object of generators** for $\mathcal{E}$ over $\mathcal{F}$ (via f) if, for any $X \in \mathcal{E}$, there exist an object $Y \in \mathcal{F}$, and a diagram

$$\begin{tikzcd} S \arrow[r, rightarrowtail] \arrow[d, twoheadrightarrow] & f^*(Y) \times G \\ X & \end{tikzcd}$$

   in $\mathcal{E}$ presenting X as a subquotient of $f^*(Y) \times G$.

2. The geometric morphism $f \colon \mathcal{E} \to \mathcal{F}$ is **bounded** if $\mathcal{E}$ has an object of generators over $\mathcal{F}$ via f.

Recall that a geometric morphism $f \colon \mathcal{E} \to \mathcal{F}$ is *localic* if f is bounded and the object of generators is the terminal object 1. In other words, localic geometric morphisms are a special case of bounded geometric morphisms.

We now recall the definition of a local map of toposes [Law86, Law89, JM89] (see in particular [JM89, Proposition 1.4]).

Definition 7.1.2. Let $\mathcal{E}$ and $\mathcal{F}$ be elementary toposes. A geometric morphism $f = (f^*, f_*) \colon \mathcal{E} \to \mathcal{F}$ is **local** if it is a bounded geometric morphism and if the direct image functor f_* has a right adjoint $f^!$ which is full and faithful.

Examples 7.1.3.

(i) Let X be a topological space, and suppose that there is a generic point $x \in X$, that is, a point x whose only neighborhood is the whole space. The space X could, *e.g.*, be a Scott domain (viewed as a topological space with the Scott topology) with the point x the least element $\perp$ of X. Then the geometric morphism $(\Delta, \Gamma) \colon \mathrm{Sh}(X) \to \mathbf{Set}$ from the topos of sheaves on X to $\mathbf{Set}$ is local. The reason is that in this case the global sections functor Γ is seen to coincide with the stalk functor $F \mapsto F_x$ (use, *e.g.*, [MM92, Section II.5, Pages 83–84]) and therefore, by [MM92, Lemma II.6.7, Page 93] it has a right adjoint (the sky-scraper sheaf functor Sky_x concentrated at x). A more abstract description of this example can be found in Section 9.2.

(ii) The geometric morphism $(\Delta, \Gamma)\colon \mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A_\sharp)$ from the relative realizability topos over A and $A_\sharp$ to the standard realizability topos over $A_\sharp$ is local, see Theorem 6.2.3.

(iii) Let $\mathbb{C}$ be a small category with finite limits and $i\colon \mathbb{D} \hookrightarrow \mathbb{C}$ a full subcategory, closed under the same. The geometric morphism $\widehat{\mathbb{C}} \to \widehat{\mathbb{D}}$ between the presheaf categories with direct image the restriction i^* along i is then a local map.

(iv) The topological topos of Johnstone [Joh79b] is local over **Set**.

(v) The "gros" topos of sheaves for the open cover topology on a suitable small subcategory of topological spaces, see [MM92, Chapter III, Section 2] is local over **Set**. Indeed the global sections functor has a left adjoint sending sets to (functors represented by) indiscrete spaces and a right adjoint defined by means of indiscrete spaces. According to Johnstone and Moerdijk [JM89], the definition of "gros topos" does not yet seem to admit a precise definition, but Lawvere [Law86] has argued that the property of being local should be part of the definition of a gros topos. We hope that our axiomatization of local maps may prove useful in establishing a suitable definition of gros topos. We remark that one may consider the exact completion $(\mathbf{Top})_{\mathrm{ex/lex}}$ of the category of topological spaces **Top** as an example of a gros non-topos which is "local" over **Set** [CR99, MS99].

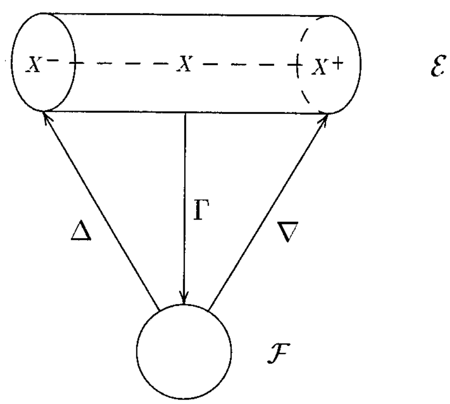
For more examples of local maps, see [ABS99, JM89] and the references therein.

Local maps of toposes have been pictured in an interesting way by Lawvere [Law86, Law89, Law91] as a so-called *adjoint cylinder* — from [Law91] we quote:

> By a level in a category of Being, I mean a ("downward") functor, from it to a smaller category which has both left and right adjoints which are full inclusions. Such a pair of categories and triple of functors is a unity-and-identity-of-opposites (UIO) in the sense that the big category unites the two opposite subcategories which in themselves are

> identical with the smaller category. One can picture the big category as a (horizontal) cylinder, some objects of which lie on the identical right or left ends. The two ends are opposite not only because we picture them so, but for the intrinsic reason of adjointness; every object in the category lies on a unique horizontal thread, two objects lying on the same thread iff the downward functor assigns to them isomorphic objects in the smaller (or lower) category.

The adjoint cylinder is pictured as follows:



(7.1)

7.2 Approach to Axiomatization

In this section we recall some background material from [KL89] and outline the approach we take to the axiomatization of local maps.

For the remainder of this section, let $\mathcal{E}$ and $\mathcal{F}$ be two elementary toposes with adjoint functors between them, as in the situation

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\Gamma"] \\ \mathcal{F} \arrow[u, bend left=60, "\Delta"] \arrow[u, bend right=60, "\nabla"'] \end{tikzcd} \qquad \Delta \dashv \Gamma \dashv \nabla, \qquad \Delta \text{ full and faithful and lex.} \tag{7.2}$$

Recall that a full subcategory $\mathbb{B}$ of a category $\mathbb{A}$ is said to be **replete** if, whenever $X \in \mathbb{B}$ and $X \cong Y$ in $\mathbb{A}$, then also $Y \in \mathbb{B}$.

By analogy to topological examples [Joh79b, Law89, Law86] in the situation in (7.2) we refer to the objects in the replete image of Δ as the **discrete objects** and to the objects in the replete image of ∇ as the **codiscrete objects**.

Note that since Δ is full and faithful we have, by [MM92, Lemma 1, Section VII.4, Page 369] (or [KL89, Proposition 2.3, Page 297]), that also ∇ is full and faithful.

In the situation above we thus have a geometric morphism $(\Gamma, \nabla)\colon \mathcal{F} \to \mathcal{E}$, whose direct image functor, ∇, is full and faithful. It follows by standard results [MM92, Corollary 7, Section VII.4, Page 375] that there is a Lawvere-Tierney topology j on $\mathcal{E}$ and an equivalence $\mathcal{F} \xrightarrow[\simeq]{e} \mathrm{Sh}_j\mathcal{E}$ such that the diagram of geometric morphisms

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "{(\Gamma,\nabla)}"] \arrow[dr, "e", "\simeq"'] & \mathcal{E} \\
& \mathrm{Sh}_j\mathcal{E} \arrow[u, hook, "i"']
\end{tikzcd}$$

commutes up to a natural isomorphism $e^*i^* \cong \Gamma$. Here $i = (i^*, i_*) = (\mathbf{a}, i)$ with $\mathbf{a}$ the associated sheaf functor and i the inclusion of sheaves. From the proof of [MM92, Corollary 7, Section VII.4, Page 375] it follows that

$$e = (e^*, e_*) = (\Gamma \circ i, \mathbf{a} \circ \nabla).$$

From the equivalence $\mathcal{F} \xrightarrow[\simeq]{e} \mathrm{Sh}_j\mathcal{E}$ it follows that the associated sheaf functor $\mathbf{a}$ has a left exact left adjoint, namely $\Delta \circ \Gamma \circ i \dashv \mathbf{a}$. Summarizing we thus have the following situation

$$\begin{tikzcd}
& \mathcal{E} \arrow[dl, bend left=10, "\Gamma"] \arrow[dr, bend left=10, "\mathbf{a}"] & \\
\mathcal{F} \arrow[ur, bend left=30, "\Delta"] \arrow[ur, bend right=10, "\nabla"'] \arrow[rr, bend left=5, "\mathbf{a}\nabla"] & & \mathrm{Sh}_j\mathcal{E} \arrow[ul, bend right=30, "i"'] \arrow[ul, bend left=10, "\Delta\Gamma i"] \arrow[ll, bend left=5, "\Gamma i"]
\end{tikzcd}
\qquad
\begin{array}{l}
\Delta \dashv \Gamma \dashv \nabla, \\
\Delta\Gamma i \dashv \mathbf{a} \dashv i, \\
\Gamma i\,\mathbf{a} \cong \Gamma, \\
\mathbf{a}\,\nabla\Gamma i \cong id, \\
\Gamma i\,\mathbf{a}\,\nabla \cong id, \\
\Delta \text{ full and faithful and lex}, \\
\nabla \text{ full and faithful.}
\end{array}
\qquad (7.3)$$

We now state a couple of conventions and then recall the notion of an essential localization from [KL89].

Convention 7.2.1. For the remainder of this chapter we will confuse $\mathcal{F}$ with its full replete image along Δ in $\mathcal{E}$. In other words, we will assume Δ is just an inclusion functor. Moreover, we abbreviate and use *subcategory* to mean a *full replete subcategory.*

Recall that a subcategory $\mathbb{B}$ of a category $\mathbb{A}$ is **reflective** if the inclusion of $\mathbb{B}$ in $\mathbb{A}$ has a left adjoint, called the **reflector**. A reflective subcategory $\mathbb{B}$ of a lex category $\mathbb{A}$ is said to be a **localization** of $\mathbb{A}$ if the reflector preserves finite limits, and a localization $\mathbb{B}$ of $\mathbb{A}$ is an **essential localization** if the reflector has a left adjoint. Thus in our situation (7.3), $\mathrm{Sh}_j\mathcal{E}$ is an essential localization of $\mathcal{E}$.

Dually, we say that a subcategory $\mathbb{B}$ of $\mathbb{A}$ is **coreflective** if the inclusion of $\mathbb{B}$ in $\mathbb{A}$ has a right adjoint, called the **coreflector**. A coreflective subcategory $\mathbb{B}$ of a category $\mathbb{A}$ with finite colimits is said to be a **colocalization** of $\mathbb{A}$ if the coreflector preserves finite colimits, and a colocalization $\mathbb{B}$ of $\mathbb{A}$ is said to be an **essential colocalization** of $\mathbb{A}$ if the coreflector has a right adjoint. Thus in our situation (7.3), $\mathcal{F}$ is an essential colocalization of $\mathcal{E}$.

We shall make use of the fact that reflective subcategories can be characterized by an orthogonality condition. Therefore we recall the following definitions and notation from [FK72], see also [KL89, Bor94a].

Let $f\colon A \to B$ be a morphism in a category $\mathbb{C}$ and let X be an object of $\mathbb{C}$. Then we say that f and X are **orthogonal** and write $f \perp X$ when $\mathbb{C}(f, X)\colon \mathbb{C}(B, X) \to \mathbb{C}(A, X)$ is a bijection, that is, if for all $a\colon A \to X$, there exists a unique $b\colon B \to X$ such that

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & X \\ B \arrow[ur, dashed, "b"'] & \end{tikzcd}$$

commutes. Moreover, we say f and X are **coorthogonal** and write $X \top f$ when $\mathbb{C}(X, f)\colon \mathbb{C}(X, A) \to \mathbb{C}(X, B)$ is a bijection, that is, if for all $b\colon X \to B$, there exists a unique $a\colon X \to A$ such that

$$\begin{tikzcd} & A \arrow[d, "f"] \\ X \arrow[ur, dashed, "a"] \arrow[r, "b"'] & B \end{tikzcd}$$

commutes.[1]

For a subcategory $\mathbb{B}$ of $\mathbb{A}$ we write $\mathbb{B}^{\perp}$ for the class of all morphisms orthogonal to every $X \in \mathbb{B}$, and given a class of morphisms $\mathcal{D}$ of a category $\mathbb{A}$, we write $\mathcal{D}^{\perp}$ for the subcategory given by those $X \in \mathbb{A}$ orthogonal to every $f \in \mathcal{D}$. Likewise, for a subcategory $\mathbb{B}$ of $\mathbb{A}$ we write $\mathbb{B}^{\top}$ for the class of all morphisms coorthogonal to every $X \in \mathbb{B}$, and given a class of morphisms

[1] Comparing with the terminology in [Bor94a], f and X are orthogonal in our sense if f is orthogonal *to* X in the sense of item (1) in Definition 5.4.2 of [Bor94a]; and f and X are coorthogonal in our sense if X is orthogonal *to* f in the sense of item (2) in Definition 5.4.2 of [Bor94a].

$\mathcal{D}$ of a category $\mathbb{A}$, we write $\mathcal{D}^{\top}$ for the subcategory given by those $X \in \mathbb{A}$ coorthogonal to every $f \in \mathcal{D}$.

Further recall that orthogonality conditions describe reflective subcategories and that, dually, coorthogonality conditions describe coreflective subcategories:

Proposition 7.2.2. *Let $\mathbb{B}$ be a reflective subcategory of $\mathbb{A}$ with reflector $R\colon \mathbb{A} \to \mathbb{B}$. Write $\mathcal{D}$ for the class of all morphisms f in $\mathbb{A}$ inverted by R (*i.e.*, for which $R(f)$ is iso). Then $\mathbb{B} = \mathcal{D}^{\perp}$ and, moreover, $\mathbb{B}^{\perp} = \mathcal{D}$.*

Proof. See [Bor94a, Proposition 5.4.4] or [KL89, Proposition 2.1]. □

Thus by Proposition 7.2.2 (see also [Joh77]), the category $\mathrm{Sh}_j\mathcal{E}$ is exactly $\mathcal{D}^{\perp}$, where $\mathcal{D}$ is the class of morphisms inverted by the associated sheaf functor $\mathbf{a}$. Moreover, by Convention 7.2.1, $\mathcal{F}$ is precisely $\mathcal{D}^{\top}$, where $\mathcal{D}$ is the class of morphisms inverted by Γ.

Following Kelly and Lawvere [KL89], an ordered pair $(\mathbb{B}, \mathbb{C})$ of subcategories of $\mathbb{A}$ is called an **associated pair (of $\mathbb{A}$)** if $\mathbb{B}$ is reflective, $\mathbb{C}$ is coreflective, and $\mathbb{B}^{\perp} = \mathbb{C}^{\top}$.

The following is (part of) Theorem 2.4 in [KL89].

Theorem 7.2.3.

1. *Let $(\mathbb{B}, \mathbb{C})$ be an associated pair of $\mathbb{A}$, with*

$$\begin{tikzcd} \mathbb{A} \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, shift left, "I"] \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} \mathbb{C} \arrow[r, shift left, "J"] \arrow[r, phantom, "\perp"] & \mathbb{A} \arrow[l, shift left, "S"] \end{tikzcd},$$

   where I and J are the inclusions. Then

   (a) *each of $\mathbb{B}$ and $\mathbb{C}$ is uniquely determined by the other, since we have $\mathbb{C} = \mathbb{B}^{\perp\top}$ and $\mathbb{B} = \mathbb{C}^{\top\perp}$;*

   (b) *the functors $SI\colon \mathbb{B} \to \mathbb{C}$ and $RJ\colon \mathbb{C} \to \mathbb{B}$ are mutually inverse equivalences;*

   (c) *$\mathbb{B}$ is an essential localization and $\mathbb{C}$ is an essential colocalization.*

2. *Moreover, every localization $\mathbb{B}$ of a category $\mathbb{A}$ forms part of an associated pair $(\mathbb{B}, \mathbb{C})$; and if $U\colon \mathbb{B} \to \mathbb{A}$ is any left adjoint of the reflector $R\colon \mathbb{A} \to \mathbb{B}$, we can describe $\mathbb{C} = \mathbb{B}^{\perp\top}$ alternatively as the full replete image of U.*

For the proof of the theorem, which uses (1) that the counit of the adjunction $R \dashv I$ is iso since I is full and faithful and (2) that the unit of the adjunction $J \dashv S$ is iso since J is full and faithful, see [KL89]. Here we just mention that in item (1c), the left adjoint to R is JSI and the right adjoint to S is IRJ. Thus given an associated pair $(\mathbb{B}, \mathbb{C})$ of $\mathbb{A}$, we have the following situation:

$$
\begin{tikzcd}
& \mathbb{A} \arrow[dl, "S"'] \arrow[dr, "R"] & \\
\mathbb{C} \arrow[ur, bend left, "J"] \arrow[ur, shift right, "IRJ"'] \arrow[rr, shift left, "RJ"] & & \mathbb{B} \arrow[ul, bend right, "I"'] \arrow[ul, shift left, "JSI"] \arrow[ll, shift left, "SI"]
\end{tikzcd}
\qquad
\begin{array}{l}
J \dashv S \dashv IRJ, \\
JSI \dashv R \dashv I, \\
SIRJ \cong id, \\
RJSI \cong id, \\
J, I \text{ full and faithful}, \\
IRJ, JSI \text{ full and faithful.}
\end{array}
\tag{7.4}
$$

Note that in (7.3), the pair $(\mathrm{Sh}_j\mathcal{E}, \mathcal{F})$ is an associated pair of $\mathcal{E}$. Moreover, by item (1a) in Theorem 7.2.3 and by Proposition 7.2.2 we have that $\mathcal{F} = (\mathrm{Sh}_j\mathcal{E})^{\perp\top} = \mathcal{D}^\top$, where $\mathcal{D}$ is the class of all morphisms inverted by $\mathbf{a}$. Thus, comparing (7.4) with the situation in (7.3), we see that from just knowing that $(\mathrm{Sh}_j\mathcal{E}, \mathcal{F})$ is an associated pair, everything in (7.3) follows, *except* that Δ (and thus $\Delta\Gamma i$) is lex.

Recall that our goal is to axiomatize when a topos $\mathcal{E}$ is local over another topos $\mathcal{F}$, as in (7.2). By the preceding discussion it is now clear that it suffices to axiomatize the situation in (7.3) and that, moreover, it suffices to show that

$$(\mathrm{Sh}_j\mathcal{E}, \mathcal{F}) \text{ is an associated pair of } \mathcal{E} \text{ and } \Delta \text{ is lex.} \tag{7.5}$$

(To be pedantic, by showing (7.5), we really show that $\mathcal{E}$ is local over the replete image of $\mathcal{F}$ along Δ in $\mathcal{E}$, see Convention 7.2.1, but from this it, of course, follows that $\mathcal{E}$ is local over $\mathcal{F}$ since $\mathcal{F}$ is equivalent to its full replete image along Δ in $\mathcal{E}$.)

We already know how to describe $\mathrm{Sh}_j\mathcal{E}$ by means of axioms on $\mathcal{E}$, namely by a Lawvere-Tierney topology j. Also note that $\mathrm{Sh}_j\mathcal{E}$ is always a reflective subcategory of $\mathcal{E}$. As explained above, $\mathcal{F} = \mathcal{D}^\top$, where $\mathcal{D}$ is the class of morphisms inverted by $\mathbf{a}$. Hence, *given* a topology j such that $\mathcal{F} \simeq \mathrm{Sh}_j\mathcal{E}$, to show (7.5), it suffices to show that

$$\mathcal{D}^\top \text{ is a coreflective subcategory of } \mathcal{E} \text{ and the inclusion } \mathcal{D}^\top \hookrightarrow \mathcal{E} \text{ is lex.} \tag{7.6}$$

This is the approach we shall take. We shall assume given a topos $\mathcal{E}$ with a topology j and then impose further axioms on $\mathcal{E}$ and j allowing us to prove (7.6). Let us emphasize that $\mathcal{D}^{\top}$ is completely well-defined given just $\mathcal{E}$ and j — $\mathcal{D}^{\top}$ is the full subcategory of coorthogonal objects to the class of all morphisms inverted by the associated sheaf functor $\mathbf{a}$, which we, of course, know exists given $\mathcal{E}$ and j.

7.3 Axiomatization

For the remainder of this section, let $\mathcal{E}$ be a topos and j a Lawvere-Tierney topology on $\mathcal{E}$. We write $\mathrm{Sh}_j\mathcal{E}$ for the full subcategory of sheaves with inclusion i and associated sheaf functor $\mathbf{a}$. Let $\mathcal{D}_j$ be the class of all morphisms inverted by $\mathbf{a}$ and define $\mathrm{D}_j\mathcal{E}$ to be $\mathcal{D}_j^{\top}$, the full subcategory of coorthogonal objects to $\mathcal{D}_j$. We refer to the objects in $\mathrm{D}_j\mathcal{E}$ as the **discrete objects** and to $\mathrm{D}_j\mathcal{E}$ as the category of discrete objects.[2]

Remark 7.3.1. As yet, we do not have a very satisfactory internal (in the internal logic of $\mathcal{E}$) definition of discrete object. The problem is that the definition above involves quantification over all objects of the topos and thus it is not straightforward to internalize it. (Of course, in the end, for a topos satisfying the axioms, we can say that an object is discrete if it is isomorphic to it's own associated discrete object, which can be found in the internal logic, but that is not very illuminating.)

Lemma 7.3.2. *The category* $\mathrm{D}_j\mathcal{E}$ *has finite colimits and the inclusion*

$$\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$$

preserves them.

Proof. Because discrete objects are coorthogonal to morphisms inverted by $\mathbf{a}$ and they only occur on the left in the definition of coorthogonal. In more detail, suppose $e\colon X \to Y$ is inverted by $\mathbf{a}$ and suppose $(C_i)_{i\in I}$ is a finite

[2] Regarding the choice of terminology: We call the objects in $\mathrm{D}_j\mathcal{E}$ "discrete" by analogy to the topological examples, as mentioned in Section 7.2. We would have liked to call the objects "cosheaves" since they are exactly the objects which are *co*orthogonal to the morphisms inverted by $\mathbf{a}$ and sheaves are the objects which are orthogonal to the morphisms inverted by $\mathbf{a}$. However, "cosheaf" has already been used to describe something else, namely a sheaf valued in an opposite category [Bun95, Ber91]. Of course, "discrete" has also been used for other concepts (*e.g.*, in the theory of realizability [HRR90]), but it seems nevertheless more innocuous to use the term "discrete" here as well. Do note, however, that in [HRR90] discrete is used to describe an orthogonal object, whereas here is it used to describe a *co*orthogonal object.

diagram of discrete objects. Then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{E}}(\varinjlim C_i, Y) &\cong \varprojlim\,(\mathrm{Hom}_{\mathcal{E}}(C_i, Y)) \\ &\cong \varprojlim\,(\mathrm{Hom}_{\mathcal{E}}(C_i, X)) \\ &\cong \mathrm{Hom}_{\mathcal{E}}(\varinjlim C_i, X),\end{aligned}$$

as required. □

Definition 7.3.3. Let j be a Lawvere-Tierney topology in a topos $\mathcal{E}$ and write $V \mapsto \overline{V}$ for the associated closure operation on subobjects $V \rightarrowtail X$. We say j is **principal** if, for all $X \in \mathcal{E}$, the closure operation on $\mathrm{Sub}(X)$ has a left adjoint $U \mapsto U^\circ$, called **interior**, that is,

$$U^\circ \leq V \iff U \leq \overline{V} \qquad \text{in } \mathrm{Sub}(X).$$

Remark 7.3.4. The interior operation is *not* assumed to commute with pullback. It follows (*e.g.*, by the fact that externalization of internal categories is a locally full and faithful 2-functor, see [Jac99]) that in general the interior operation is not induced by an internal map on the subobject classifier Ω in the topos $\mathcal{E}$.

Lemma 7.3.5. *A topology j in a topos $\mathcal{E}$ is principal iff, for all $X \in \mathcal{E}$, there exists a* least dense *subobject U_X of X.*

Proof. Suppose j is principal. Let $X \in \mathcal{E}$ and let $U_X = X^\circ$. By the unit of the adjunction, we have $X \leq \overline{X^\circ}$ in $\mathrm{Sub}(X)$, so $\overline{U_X} = X$ and U_X is dense. Suppose $V \in \mathrm{Sub}(X)$ is dense. Then $X \leq \overline{V}$, so by adjointness $U_X \leq V$, as required.

For the other direction, suppose U_X is the least dense subobject of $X \in \mathcal{E}$. Write X° for U_X. To show the required adjunction it suffices to show (1) that $\overline{V}^\circ \leq V$ and (2) that $\overline{V^\circ} \geq V$ (from which we get the counit and the unit). Write $V \leq^d X$ to denote that V is a dense subobject of X.

For (1), note that

$$\begin{array}{rll} & V \leq^d \overline{V} & \text{always true} \\ \Longrightarrow & \overline{V}^\circ \leq V & \text{since } \overline{V}^\circ \text{ is least dense in } \overline{V}. \end{array}$$

For (2), note that

$$\begin{array}{rll} & V^\circ \leq^d V & \text{by definition} \\ \Longrightarrow & V \leq \overline{V^\circ} & \text{by density (in fact, } \overline{V^\circ} = V). \end{array}$$

This completes the proof of the lemma. □

Suppose given a topos $\mathcal{E}$ with a principal topology j. Then by the lemma, for all $X \in \mathcal{E}$ there exists a least dense subobject U_X in $\mathrm{Sub}(X)$. We now show that the operation $X \mapsto U_X$ extends to a functor on $\mathcal{E}$. To this end, suppose $f: X \to Y$ and consider the following diagram

$$\begin{tikzcd} U_X \arrow[r, dashed] \arrow[dr] & f^*(U_Y) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & U_Y \arrow[d, tail] \\ & X \arrow[r] & Y, \end{tikzcd}$$

where the right hand square is a pullback. Now note that, since j-closure commutes with pullback, we have that $\overline{f^*(U_Y)} = f^*(\overline{U_Y}) = f^*Y = X$, so $f^*(U_Y)$ is dense in X, and thus $U_X \leq f^*(U_Y)$ in $\mathrm{Sub}(X)$. Hence there is an arrow $U_X \to f^*(U_Y)$ as shown in the diagram above. Letting $U(f)$ be the composite arrow across the top in the diagram above, we clearly get a functor $U: \mathcal{E} \to \mathcal{E}$.

We now show that functor U is idempotent. Write $V \leq^d X$ to denote that V is a dense subobject of X. By definition, $U(U(X)) \leq^d U(X)$ and $U(X) \leq^d X$, from which it follows[3] that $U(U(X)) \leq^d X$. Hence $U(U(X)) \geq U(X)$ since $U(X)$ is the *least* dense subobject in X. Thus $U(U(X)) = U(X)$ and U is idempotent.

We write $X \mapsto X^\circ : \mathcal{E} \to \mathcal{E}$ for the functor U. We refer to this functor as the **interior** functor. (Note that this notation and terminology is in accordance with Definition 7.3.3.)

Remark 7.3.6. Expressed using the internal logic of $\mathcal{E}$, a Lawvere-Tierney topology $j: \Omega \to \Omega$ in $\mathcal{E}$ is principal if, for each type X, there is an atomic predicate $U_X : X \to \Omega$ satisfying the following axiom and rule (the rule is a scheme, for any predicate $\varphi: X \to \Omega$ on X):

$$\frac{}{\vdash \forall x : X.\ j(U_X(x))}\ \textbf{dense}$$

$$\frac{\vdash \forall x : X.\ j(\varphi(x))}{\vdash \forall x : X.\ U_X(x) \supset \varphi(x)}\ \textbf{least dense}$$

Using the atomic predicates, the interior functor $X \mapsto X^\circ$ can be defined as $X \mapsto \{ x : X \mid U_X \}$ (where $\{ x : X \mid U_X \}$ of course denotes the subset

[3] In a topos with a Lawvere-Tierney topology j, we always have that if U is a dense subobject of V and V is a dense subobject of W, then U is a dense subobject of W, see, *e.g.*, [Jac99, Exercise 5.6.2(iii)].

type of X given by the predicate U_X). This defines a functor because if $f\colon X \to Y$, then $\forall x\colon X.\ U_X(x) \supset U_Y(f(x))$, because by the least-dense rule it suffices to show that $\forall x\colon X.\ j(U_Y(f(x))$ is valid in $\mathcal{E}$, but this holds since by the density axiom, $\forall y\colon Y.\ j(U_y(y))$. Formally, f° is thus $\lambda x.\ \mathsf{i}(f(\mathsf{o}(x)))$, where i and o are the injections and projections associated with the subset types. We shall leave out those injections and projections in the following, when giving further examples of reasoning using the internal language.

Lemma 7.3.7. *The interior functor $X \mapsto X^\circ\colon \mathcal{E} \to \mathcal{E}$ preserves monomorphisms.*

Proof. For a mono $m\colon X \rightarrowtail Y$, we get the following diagram in $\mathcal{E}$

$$\begin{tikzcd}
X \arrow[r, tail] & Y \\
X^\circ \arrow[u, tail] \arrow[r, "m^\circ"'] & Y^\circ \arrow[u, tail]
\end{tikzcd}$$

from which it follows that m° is monic (since postcomposing with $X^\circ \rightarrowtail X$ gives a monic). □

For future use, we record the following easy corollary of Lemma 7.3.5.

Corollary 7.3.8. *Let $\mathcal{E}$ be a topos with a principal topology j. Then we have that, for all $X \in \mathcal{E}$, for all $V \in \mathrm{Sub}(X)$,*

$$\overline{V^\circ} = \overline{V} \quad \textit{and} \quad \overline{V}^\circ = V^\circ \qquad \textit{in } \mathrm{Sub}(X).$$

The following lemma says that interior commutes with taking images (existential quantification). For $f\colon X \to Y$ in $\mathcal{E}$ we write $\exists_f$ for the left adjoint to the pullback functor $f^*\colon \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$.

Lemma 7.3.9. *Let $\mathcal{E}$ be a topos with a principal topology j. Let $X, Y \in \mathcal{E}$, $V \in \mathrm{Sub}(X)$, and $f\colon X \to Y$ in $\mathcal{E}$. Then $\exists_f(V^\circ) \cong (\exists_f V^\circ)$.*

Proof. Because it holds for the right adjoints, $f^*(\overline{V}) \cong (\overline{f^*V})$ [MM92]. □

Remark 7.3.10. We rephrase the above lemma using the internal language of $\mathcal{E}$. Let $f\colon X \to Y$ in $\mathcal{E}$ and let $Z = \{\, y\colon Y \mid \exists x\colon X.\ f(x) =_Y y \,\}$. Then

$$\vdash \forall y\colon Z.\ U_Z(y) \supset\subset \exists x\colon X.\ f(x) =_Y y \wedge U_X(x) \tag{7.7}$$

is valid in $\mathcal{E}$.

Lemma 7.3.11. *Let $\mathcal{E}$ be a topos with a principal topology j. The interior functor $X \mapsto X^\circ \colon \mathcal{E} \to \mathcal{E}$ preserves epis.*

Proof. Let $f \colon X \twoheadrightarrow Y$ be an epi and let $m \colon X^\circ \rightarrowtail X$ be the interior of X. Since f is epic, the image $\mathrm{Im}(f)$ of f equals Y. By Lemma 7.3.9, we get that $\mathrm{Im}(fm) = \exists_f(X^\circ) = (\exists_f X)^\circ = \mathrm{Im}(f)^\circ = Y^\circ$. Write

$$X^\circ \overset{e}{\twoheadrightarrow} \mathrm{Im}(fm) \overset{n}{\rightarrowtail} Y$$

for the image factorization of fm. Thus there is an epic $e \colon X^\circ \twoheadrightarrow \mathrm{Im}(fm) = Y^\circ$ and it just remains to verify that this epic indeed is f°. To this end, consider the diagram defining $f^\circ = hg$:

$$\begin{tikzcd}
X^\circ \arrow[r, tail, "g"] \arrow[dr, "m"'] & f^*(Y^\circ) \arrow[r, "h"] \arrow[d, tail] & Y^\circ = \mathrm{Im}(fm) \arrow[d, tail, "n"] \\
& X \arrow[r, two heads, "f"'] & Y.
\end{tikzcd}$$

Now $ne = fm = nhg$, so $e = hg = f^\circ$ since n is monic, completing the proof of the lemma. □

Remark 7.3.12. The above lemma can also easily be proved in the internal language: Suppose $f \colon X \to Y$ is epi, *i.e.*, $\forall y \colon Y.\ \exists x \colon X.\ f(x) = y$. We are to show that

$$\forall y \colon \{\, y \colon Y \mid U_Y(y) \,\}.\ \exists x \colon \{\, x \colon X \mid U_X(x) \,\}.\ f(x) = y.$$

It suffices to show that

$$\forall y \colon Y.\ U_Y(y) \supset \exists x \colon X.\ f(x) = y \wedge U_X(x). \tag{7.8}$$

But, letting $Z' = \mathrm{Im}(f)$ we have that $Z' = Y$, as f is epi, so (7.8) is equivalent to

$$\forall y \colon Z'.\ U_{Z'}(y) \supset \exists x \colon X.\ f(x) = y \wedge U_X(x),$$

which holds by Remark 7.3.10.

Definition 7.3.13. Let $\mathcal{E}$ be a topos with a principal topology j. We then say that $X \in \mathcal{E}$ is **open** if $X^\circ \cong X$.

Remark 7.3.14. Phrased using the internal language, an object X is open if and only if $\forall x : X.\ U_X(x)$ holds. The reader is warned against confusing this definition of open with the topological notion of an open set (in the same way as one should not confuse the closure operation associated with a Lawvere-Tierney topology with topological closure [Joh77, Page 78]).

In the following development the open objects play a role similar to the role separated objects play for sheaves. Indeed, just like every sheaf is separated, we have that every discrete object is open:

Lemma 7.3.15. *Let $\mathcal{E}$ be a topos with a principal topology j. Then every discrete object $C \in \mathrm{D}_j\mathcal{E}$ is open.*

Proof. Since $m \colon C^\circ \rightarrowtail C$ is dense, we have that $\mathbf{a}(m)$ is iso (again using [Joh77, Proposition 3.42]). Thus, since C is discrete, there is a unique lift of the identity across m as in

$$\begin{tikzcd} & C^\circ \arrow[d, tail, "m"] \\ C \arrow[ur, dashed, "\overline{id}"] \arrow[r, "id"'] & C, \end{tikzcd}$$

from which we see that $C \cong C^\circ$. $\square$

We have the following alternative characterization of openness.

Lemma 7.3.16. *Let $\mathcal{E}$ be a topos with a principal topology j. Let $X \in \mathcal{E}$ and write $\Delta_X : X \rightarrowtail X \times X$ for the diagonal. Then X is open if and only if ${\Delta_X}^\circ = \Delta_X$ in $\mathrm{Sub}(X \times X)$.*

Proof. Consider the diagram

$$\begin{tikzcd} X^\circ \arrow[rr, tail] \arrow[dr, "{\Delta_X}^\circ"'] & & X \arrow[dl, "\Delta_X"] \\ & X \times X. & \end{tikzcd}$$

Now X is open iff $X^\circ \cong X$ iff ${\Delta_X}^\circ = \Delta_X$ in $\mathrm{Sub}(X \times X)$. (The point is just that the domain of Δ_X is X.) $\square$

Lemma 7.3.17. *Let $\mathcal{E}$ be a topos with a principal topology j. Then a quotient of an open object is open.*

Proof. Suppose X is open and that $e\colon X \twoheadrightarrow Y$ is a quotient of X. Then by Lemma 7.3.9 and using the assumption that $X \cong X^\circ$ we have that

$$Y \cong \mathrm{Im}(e) = \exists_e X \cong \exists_e(X^\circ) \cong (\exists_e X)^\circ \cong (\exists_e(X^\circ))^\circ,$$

so $\exists_e(X^\circ)$ is open, so Y is open. □

Remark 7.3.18. In the internal language the argument goes as follows. Suppose X is open and that $e\colon X \twoheadrightarrow Y$ is a quotient map. Since X is open, *i.e.*, $\forall x.\ U_X(x)$, and e is epic, we have $\forall y\colon Y.\ \exists x\colon X.\ f(x)\colon y \wedge U_X(x)$. By Remark 7.3.10 and using that e is epic so that the image of f equals Y, we have that $\forall y\colon Y.\ U_Y(y) \supset\subset \exists x\colon X.\ f(x) = y \wedge U_X(x)$. Combining the properties we then get that $\forall y\colon Y.\ U_Y(y)$, so Y is open.

Definition 7.3.19. Let $\mathcal{E}$ be a topos with a principal topology j. We define $\mathrm{O}_j\mathcal{E}$ to be the full subcategory of $\mathcal{E}$ of open objects.

Given a topos $\mathcal{E}$ with a topology j, the category of separated objects is a reflective subcategory of $\mathcal{E}$. Analogously, we here find that (for a principal topology) the category of open objects is a *co*reflective subcategory of $\mathcal{E}$.

Lemma 7.3.20. *Let $\mathcal{E}$ be a topos with a principal topology j. Then $\mathrm{O}_j\mathcal{E}$ is a coreflective subcategory of $\mathcal{E}$.*

Proof. The functor $X \mapsto X^\circ$ is right adjoint to the inclusion of $\mathrm{O}_j\mathcal{E}$ into $\mathcal{E}$. □

As mentioned before, given a topos $\mathcal{E}$ with topology j, an object X is a sheaf iff X is orthogonal to the class of all morphisms inverted by the associated sheaf functor $\mathbf{a}$. Recall that one does not need to consider orthogonality with respect to all morphisms inverted by $\mathbf{a}$, but can restrict attention to dense monos—indeed, the usual definition of a sheaf just requires orthogonality with respect to dense monos [MM92]. We shall now show that also in the case of discrete objects, we need not require coorthogonality with respect to *all* morphisms inverted by $\mathbf{a}$ but just with respect to a smaller class of what we shall call *codense epis*.

Definition 7.3.21. Let $\mathcal{E}$ be a topos with a principal topology j and let $e\colon X \twoheadrightarrow Y$ be an epi. Write $\Delta_X \rightarrowtail X \times X$ for the diagonal and write K_e for the kernel of e, viewed as a subobject of $X \times X$. We say that e is **codense** if ${K_e}^\circ = {\Delta_X}^\circ$ in $\mathrm{Sub}(X \times X)$.

Lemma 7.3.22. *Let $\mathcal{E}$ be a topos with a principal topology j and write* $\mathbf{a}$ *for the associated sheaf functor. Then an epi $e\colon X \twoheadrightarrow Y$ is codense iff* $\mathbf{a}(e)$ *is iso.*

Proof. Suppose that $\mathbf{a}(e)$ is iso. Consider the kernel pair of e

$$K_e \underset{k'}{\overset{k}{\rightrightarrows}} X \xrightarrow{e} Y.$$

as a subobject of $X \times X$

$$K_e \xrightarrow{\langle k,k'\rangle} X \times X.$$

We are to show that $K_e{}^\circ = \Delta_X{}^\circ$ in $\mathrm{Sub}(X \times X)$. We immediately have that $K_e \geq \Delta_X$ because the kernel pair of a morphism is always an equivalence relation. Hence, by functoriality of interior, $K_e{}^\circ \geq \Delta_X{}^\circ$. It remains to show that $K_e{}^\circ \leq \Delta_X{}^\circ$. By adjointness and Corollary 7.3.8 this is equivalent to $K_e \leq \overline{\Delta_X{}^\circ} = \overline{\Delta_X}$. But $K_e \leq \overline{K_e}$, so it suffices to show that $\overline{K_e} \leq \overline{\Delta_X}$. Recall [MM92, Corollary 8, Section V.4, Page 233] that $\mathbf{a}$ induces an isomorphism $\mathrm{ClSub}_j(E) \cong \mathrm{Sub}_{\mathrm{Sh}_j\mathcal{E}}(\mathbf{a}\,E)$ between j-closed subobjects of E and subsheaves of $\mathbf{a}\,E$. Hence it suffices to show that

$$\mathbf{a}(K_e) \leq \mathbf{a}(\Delta_X) \text{ in } \mathrm{Sub}_{\mathrm{Sh}_j\mathcal{E}}(\mathbf{a}\,X \times \mathbf{a}\,X).$$

Now since $\mathbf{a}$ is lex, $\mathbf{a}$ preserves kernel pairs, so

$$\mathbf{a}\,K_e = K_{\mathbf{a}\,e} \rightrightarrows \mathbf{a}\,X \xrightarrow[\cong]{\mathbf{a}\,e} \mathbf{a}\,Y.$$

is a kernel pair. But $\mathbf{a}\,e$ is iso by assumption, hence monic, so $\mathbf{a}\,K_e = K_{\mathbf{a}\,e} = \Delta_{\mathbf{a}\,X}$, so we have the required.

For the other direction, suppose $e\colon X \twoheadrightarrow Y$ is a codense epi. We are to show that $\mathbf{a}\,e$ is iso. Since $\mathbf{a}$ as a left adjoint preserves epis, $\mathbf{a}\,e$ is epic, so it suffices to show that $\mathbf{a}\,e$ is monic. Let K_e be the kernel of e viewed as a subobject of $X \times X$. Since e is codense we get the following series of implications

$$\begin{array}{llr} & K_e{}^\circ = \Delta_X{}^\circ & \text{by definition} \\ \Longrightarrow & \overline{K_e{}^\circ} = \overline{\Delta_X{}^\circ} & \\ \Longrightarrow & \overline{K_e} = \overline{\Delta_X} & \text{by Corollary 7.3.8} \\ \Longrightarrow & K_{\mathbf{a}\,e} = \mathbf{a}\,K_e = \mathbf{a}(\Delta_X) = \Delta_{aX} & (*) \end{array}$$

where the implication $(*)$ follows since $\mathbf{a}$ is lex and

$$\mathrm{ClSub}_{\mathcal{E}}(E) \cong \mathrm{Sub}_{\mathrm{Sh}_j\mathcal{E}}(\mathbf{a}\,E),$$

see above. Since the kernel of $\mathbf{a}\,e$ is the diagonal, we conclude that $\mathbf{a}\,e$ is monic, completing the proof of the proposition. $\square$

Remark 7.3.23. Recall [Joh77, Definition 3.41] that a morphism $f\colon X \to Y$ is **almost monic** if the diagonal $\Delta_X\colon X \rightarrowtail X \times X$ is dense in the kernel of f. Moreover, for an epimorphism e, the morphism $\mathbf{a}(e)$ is iso iff e is is almost monic [Joh77, Corollary 3.43]. It thus follows immediately by the above lemma that a codense epi is an epi which is almost monic. In the internal logic, a morphism $e\colon X \to Y$ is almost monic iff $\forall x, x'\colon X.\ e(x) = e(x') \supset j(x = x')$ is valid (this description follows from the fact that e is almost monic iff e is internally injective in the fibration of j-closed subobjects [Jac99, Page 357–358]; using the description of the closed subobject fibration in *loc. cit.* one arrives at the here given description in terms of the internal logic of $\mathcal{E}$).

Remark 7.3.24. We note that just like the pullback of a dense mono is again a dense mono, it is easy to see, using the previous lemma, that the pushout of a codense epi is again a codense epi.

Now follows the promised proposition which allows to determine whether an object is discrete just by testing for coorthogonality with respect to codense epis.

Proposition 7.3.25. *Let $\mathcal{E}$ be a topos with a principal topology j. Then C is discrete if and only if C is coorthogonal to the class all codense epis in $\mathcal{E}$.*

To prove the proposition we shall make use of the following lemma.

Lemma 7.3.26. *Let $\mathcal{E}$ be a topos with a principal topology j. Suppose $C \in \mathcal{E}$ and that C is coorthogonal to the class all codense epis in $\mathcal{E}$. Then, for all dense subobjects $m\colon Y \rightarrowtail X$ and all morphisms $f\colon C \to X$, there exists a unique $f'\colon C \to Y$ such that $mf' = f$, as in*

$$\begin{tikzcd} & Y \arrow[d, tail, "m"] \\ C \arrow[ur, dashed, "f'"] \arrow[r, "f"'] & X. \end{tikzcd}$$

Proof. Let C, $m\colon Y \rightarrowtail X$, and $f\colon C \to X$ be as in the lemma. Consider the following diagrams

$$\begin{tikzcd}
 & Y \arrow[d, tail, "m"] & & \mathbf{a}\,Y \arrow[d, tail, "\mathbf{a}\,m,\,\cong"] \\
C \arrow[ur, dashed, "f'"] \arrow[r, "f"'] & X \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] & & \mathbf{a}\,X \arrow[d, shift right, "\mathbf{a}\,u"'] \arrow[d, shift left, "\mathbf{a}\,v"] \\
 & W \arrow[d, "e"] & & \mathbf{a}\,W \arrow[d, "\mathbf{a}\,e"] \\
 & P & & \mathbf{a}\,P,
\end{tikzcd}$$

where u, v is the cokernel pair of m and e is the coequalizer of u, v. Since $\mathbf{a}$ is a left adjoint, it preserves cokernel pairs and coequalizers, so $\mathbf{a}\,u, \mathbf{a}\,v$ is the cokernel pair of $\mathbf{a}\,m$, which is an iso by assumption that m is dense. Hence $\mathbf{a}\,u = \mathbf{a}\,v$. Therefore $\mathbf{a}\,e$, the coequalizers of $\mathbf{a}\,u, \mathbf{a}\,v$ is an iso and thus, by Lemma 7.3.22 e is codense. Since $euf = evf\colon C \to P$ and since $C \top e$ by assumption, we get that $uf = vf$ by uniqueness. Hence f factors uniquely through the equalizer of u, v. But m is the equalizer of u, v (as every mono in a topos is the equalizer of its cokernel pair, see [Joh77, 1.28]), so f factors uniquely through m via an f' as shown in the diagram above. □

Proof of Proposition 7.3.25. We only need to show the right-to-left implication; the other is trivial by Lemma 7.3.22. Suppose $C \in \mathcal{E}$ and that $C \top e$ for all codense epis e. We are to show that $C \top h$ for all morphisms h such that $\mathbf{a}(h)$ is iso. So suppose $h\colon X \to Y$ is such that $\mathbf{a}(h)$ is iso and let $f\colon C \to Y$ be arbitrary. Consider the following diagrams

$$\begin{tikzcd}
 & X \arrow[d, "e"] \arrow[dd, bend left, "h"] & & \mathbf{a}\,X \arrow[d, "\mathbf{a}\,e"'] \arrow[dd, bend left, "{\mathbf{a}\,h,\,\cong}"] \\
 & I \arrow[d, tail, "m"'] & & \mathbf{a}\,I \arrow[d, tail, "\mathbf{a}\,m"'] \\
C \arrow[uur, dashed, "f''"] \arrow[ur, dashed, "f'"'] \arrow[r, "f"'] & Y & & \mathbf{a}\,Y,
\end{tikzcd}$$

where me is the image factorization of h and the diagram on the right is $\mathbf{a}$ applied to this image factorization. Since $\mathbf{a}$ preserves image factorizations and $\mathbf{a}\,h$ is iso by assumption, we have that $\mathbf{a}\,m$ is iso and that $\mathbf{a}\,e$ is iso. Hence m is dense and, by Lemma 7.3.22, e is codense. Thus by Lemma 7.3.26, there exists a unique $f'\colon C \to I$ such that $mf' = f$. By assumption $C \top e$, so there exists a unique $f''\colon C \to X$ such that $ef'' = f'$. The morphism f'' is the required unique morphism showing that $C \top h$. $\square$

We now define an operation, called *exterior*, on quotients, which one can think of as a dual operation to the traditional closure operation on subobjects.

Definition 7.3.27. Let $\mathcal{E}$ be a topos with a principal topology j. For an epi $e\colon X \twoheadrightarrow Y$, we define the **exterior** of e, written $\widetilde{e}\colon X \twoheadrightarrow \widetilde{Y}$, to be the coequalizer of the interior $K_e{}^\circ$ of the kernel pair K_e of e.

Expressed in a diagram the definition looks like:

$$\begin{tikzcd}
K_e \arrow[dr, shift left, "k"] \arrow[dr, shift right, "k'"'] & & \widetilde{Y} = \mathrm{CoEq}(km, k'm) \arrow[dd, dashed, "h"] \\
& X \arrow[ur, two heads, "\widetilde{e}"] \arrow[dr, two heads, "e"'] & \\
K_e{}^\circ \arrow[uu, tail, "m"] \arrow[ur, shift left, "km"] \arrow[ur, shift right, "k'm"'] & & Y
\end{tikzcd} \tag{7.9}$$

By the universal property of the coequalizer, since $ekm = ek'm$, there is a unique map $h\colon \widetilde{Y} \to Y$ such that $h\widetilde{e} = e$, as shown in the diagram. Since e is epic, h is also epic.

Lemma 7.3.28. *Referring to the diagram* (7.9) *above, the epi h is codense.*

Proof. By Lemma 7.3.22 it suffices to show that $\mathbf{a}\,h$ is iso. Apply $\mathbf{a}$ to the diagram (7.9). Since $m\colon K_e{}^\circ \rightarrowtail K_e$ is dense, $\mathbf{a}(m)$ is iso. Hence, since $\mathbf{a}$ preserves kernel pairs and coequalizers, $\mathbf{a}\,h$ is iso. $\square$

Clearly, the exterior operation $e \mapsto \widetilde{e}$ on epis $e\colon X \twoheadrightarrow Y$ induces a well-defined operation on quotients of X. By definition of the ordering of quotients of X [AHS90, Page 113], the quotient represented by $\widetilde{e}$ is greater than the quotient represented by e. In fact, it can easily be verified that the exterior operation induces a functor $\widetilde{-}\colon \mathrm{Quot}(X) \to \mathrm{Quot}(X)$ on the quotients of X.

Lemma 7.3.29. *Let $\mathcal{E}$ be a topos with a principal topology j. For any $X \in \mathcal{E}$, the exterior functor* $\mathrm{Quot}(X) \to \mathrm{Quot}(X)$ *is idempotent.*

Proof. Consider the following diagram

$$
\begin{tikzcd}
K_e^{\circ\circ} \arrow[r, equal] \arrow[d, dashed, "d=c^\circ"'] & K_e^{\circ} \arrow[d, dashed, "c"] \arrow[dr, "b"] & \\
K_{\tilde{e}}^{\circ} \arrow[r, tail, "a"'] & K_{\tilde{e}} \arrow[dr, shift left, "n_2"] \arrow[dr, shift right, "n_1"'] & K_e \arrow[d, shift right, "m_1"'] \arrow[d, shift left, "m_2"] \\
 & & X \arrow[dll, two heads, "\tilde{\tilde{e}}"'] \arrow[dl, two heads, "\tilde{e}"'] \arrow[d, two heads, "e"] \\
\tilde{\tilde{Y}} \arrow[r, two heads, "\overline{a}"'] & \tilde{Y} \arrow[r, two heads, "\overline{b}"'] \arrow[l, dotted, two heads, bend left, "\tilde{d}"] & Y,
\end{tikzcd}
$$

where $K_e \overset{m_1}{\underset{m_2}{\rightrightarrows}} X$ is the kernel of e; $\tilde{e}$ is the coequalizer of $m_1 b, m_2 b$; and $K_{\tilde{e}} \overset{n_1}{\underset{n_2}{\rightrightarrows}} X$ is the kernel of $\tilde{e}$. By the universal property of this kernel pair, there exists a unique $c\colon K_e{}^\circ \to K_{\tilde{e}}$ such that $m_1 b = n_1 c$ and $m_2 b = n_2 c$. Now let $a\colon K_{\tilde{e}}{}^\circ \rightarrowtail K_{\tilde{e}}$ be the interior and let $d = c^\circ$ as shown in the diagram. Since interior is idempotent, $K_e{}^{\circ\circ} = K_e{}^\circ$, and the top left square commutes. Let $\tilde{\tilde{e}}$ be the coequalizer of $n_1 a, n_2 a$. Then $\tilde{\tilde{e}}$ also coequalizes $m_1 b, m_2 b$ so there exists a unique $\tilde{d}\colon \tilde{Y} \to \tilde{\tilde{Y}}$, proving that $\tilde{Y} \geq \tilde{\tilde{Y}}$ in $\mathrm{Quot}(X)$. Since we already have that $\tilde{\tilde{Y}} \geq \tilde{Y}$, we conclude that $\tilde{Y} = \tilde{\tilde{Y}}$ in $\mathrm{Quot}(X)$, as required. □

Recall that for a *dense* subobject $X \rightarrowtail Y$, the closure $\overline{X}$ of X is Y. We have a similar property for codense epis and the exterior operation:

Lemma 7.3.30. *Let $\mathcal{E}$ be a topos with a principal topology j. For any quotient $e\colon X \twoheadrightarrow Y$ for which e is codense, the exterior $\tilde{Y}$ of Y is equal to X, as quotients of X.*

Proof. Consider the diagram (7.9). Since e is codense, we have by definition that $K_e{}^\circ = \Delta_X{}^\circ$ as subobjects of $\mathrm{Sub}(X \times X)$. Hence $\tilde{Y}$ is isomorphic to the coequalizer of the diagonal and thus isomorphic to X. □

By Lemmas 7.3.28 and 7.3.30 it follows that if $e\colon X \twoheadrightarrow Y$ is a quotient, then the exterior of the induced quotient $h\colon \tilde{Y} \twoheadrightarrow Y$ is equal to $\tilde{Y}$ (as quotients of $\tilde{Y}$).

We can now finally state some conditions under which we can prove that the category of discrete objects is coreflective in the ambient topos $\mathcal{E}$. For simplicity, and because its an important example, we first consider the axiomatization of *localic* local maps in Subsection 7.3.1. In the following Subsection 7.3.2 we consider the axiomatization of arbitrary (bounded) local maps. See [Joh77, Joh81] for more on bounded and localic geometric morphisms. One may think of the localic and boundedness conditions as size-conditions, expressing that $\mathcal{E}$ is generated over $\mathrm{D}_j\mathcal{E}$, in the sense that $\mathcal{E}$ is the category $\mathrm{D}_j\mathcal{E}$-valued sheaves on an internal site in $\mathrm{D}_j\mathcal{E}$.

7.3.1 Axioms for Localic Local Maps

For $\mathcal{E}$ an elementary topos with a topology j we suggest the following **axioms for localic local maps.**

Axiom 1 j is principal.

Axiom 2 For all $X \in \mathcal{E}$, there exists a discrete object C and a diagram

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & C \\ X & \end{tikzcd}$$

in $\mathcal{E}$, presenting X as a subquotient of C.

Axiom 3 For all discrete $C \in \mathcal{E}$, if $X \rightarrowtail C$ is open, then X is also discrete.

Axiom 4 For all discrete $C, C' \in \mathcal{E}$, $C \times C'$ is discrete.

Completeness

Theorem 7.3.31. *Let $\mathcal{E}$ be a topos with a topology j and suppose that $\mathcal{E}$ and j satisfy Axioms 1–4 for localic local maps. Then the category of discrete objects $\mathrm{D}_j\mathcal{E}$ is coreflective in $\mathcal{E}$.*

Proof. We show how to construct an associated discrete object for any object $X \in \mathcal{E}$. By Axiom 2, we have a diagram

$$\begin{tikzcd} S \arrow[r, tail, "m"] \arrow[d, two heads, "e"] & C_X \\ X & \end{tikzcd}$$

in $\mathcal{E}$ presenting X as a subquotient of a discrete object C_X. Now consider the following diagram

$$\begin{tikzcd}
K_{e^\circ}{}^\circ \arrow[r, tail] & K_{e^\circ} \arrow[d, shift left] \arrow[d, shift right] & & \\
& S^\circ \arrow[r, tail, "m"] \arrow[dl] \arrow[d, "e^\circ"] & S \arrow[r, tail] \arrow[d, "e"] & C \\
\widetilde{X^\circ} \arrow[r, two heads, "h"'] & X^\circ \arrow[r, tail] & X &
\end{tikzcd}$$

Since interior preserves epimorphisms by Lemma 7.3.11, $e^\circ\colon S^\circ \to X^\circ$ is epic. The exterior $\widetilde{X^\circ}$ of the interior X° of X is obtained in the standard way, as the coequalizer of the interior $K_{e^\circ}{}^\circ$ of the kernel pair K_{e° of e°. By Axiom 3, S° is discrete and thus also $K_{e^\circ}{}^\circ$ is discrete by Axioms 3 and 4. Hence $\widetilde{X^\circ}$ is obtained as the coequalizer

$$K_{e^\circ}{}^\circ \rightrightarrows S^\circ \twoheadrightarrow \widetilde{X^\circ}$$

of a diagram of discrete objects and thus, by Lemma 7.3.2, $\widetilde{X^\circ}$ is discrete. We claim that $\widetilde{X^\circ} \twoheadrightarrow X^\circ \rightarrowtail X$ is couniversal among arrows from discrete objects into X, thus establishing the existence of a right adjoint to the inclusion $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$. Indeed, let C be any discrete object and let $f\colon C \to X$ be arbitrary. Consider the following diagram

$$\begin{tikzcd}
\widetilde{X^\circ} \arrow[r, dashed, "h"] & X^\circ \arrow[r, tail] & X \\
C \arrow[u, dashed, "f''"] \arrow[ur, dashed, "f'"] \arrow[urr, "f"'] & &
\end{tikzcd}$$

Since the interior functor $-^\circ\colon \mathcal{E} \to \mathrm{O}_j\mathcal{E}$ is right adjoint to the inclusion of open objects into $\mathcal{E}$ by Lemma 7.3.20, there is a unique morphism f' making the right triangle commute. Then since h is a codense epi by Lemma 7.3.28 and C is discrete, we have by Proposition 7.3.25 that $C \top h$, so there exists a unique f'' making the left triangle commute. This shows the required couniversality, completing the proof of the theorem. □

Corollary 7.3.32. *Let $\mathcal{E}$ be a topos with a topology j and suppose that $\mathcal{E}$ and j satisfy Axioms 1–4 for localic local maps. Then the category of discrete objects $\mathrm{D}_j\mathcal{E}$ is equivalent to $\mathrm{Sh}_j\mathcal{E}$ and the associated sheaf functor $\mathbf{a}\colon \mathcal{E} \to \mathrm{Sh}_j\mathcal{E}$ has a left adjoint.*

Proof. By Theorem 7.3.31 and the discussion in Section 7.2. □

Theorem 7.3.33. *Let $\mathcal{E}$ be a topos with a topology j and suppose that $\mathcal{E}$ and j satisfy Axioms 1–4 for localic local maps. Then the inclusion $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ is left exact and finite limits in $\mathrm{D}_j\mathcal{E}$ are computed as in $\mathcal{E}$.*

To prove the theorem it is useful to name the inclusion functor and the coreflector, as follows:

$$\begin{tikzcd} \mathrm{D}_j\mathcal{E} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift left, "R"] \end{tikzcd},$$

where

$$L \dashv R \qquad \text{and} \qquad R \circ L \cong id.$$

Thus L is the inclusion of discrete objects and R is the associated discrete functor. Recall that R is a known to have right adjoint, by Corollary 7.3.32.

The proof of the theorem proceeds by a series of lemmas. The main tool is the following lemma, which seems to be folklore (a related argument is in [Fre72, 2.61]).

Lemma 7.3.34. *Let $\mathcal{E}$ and $\mathcal{F}$ be toposes and suppose the functor $F\colon \mathcal{E} \to \mathcal{F}$ preserves finite products, monomorphisms, and pushouts. Then F is left exact.*

By this lemma and the following (which, of course, is stated under the assumptions of Theorem 7.3.33),

Lemma 7.3.35. *The functor $LR\colon \mathcal{E} \to \mathcal{E}$ preserves finite products, monomorphisms, and all colimits.*

we can then conclude that

Corollary 7.3.36. *The functor $LR\colon \mathcal{E} \to \mathcal{E}$ is left exact.*

Using this fact, we can complete the proof of Theorem 7.3.33:

Lemma 7.3.37. *The functor $L\colon \mathrm{D}_j\mathcal{E} \to \mathcal{E}$ is left exact and finite limits in $\mathrm{D}_j\mathcal{E}$ are computed as in $\mathcal{E}$.*

Let us now proceed with the proofs of the above mentioned lemmas.

Proof of Lemma 7.3.34. We need to show that F preserves equalizers, for which we use the fact that

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

is an equalizer if and only if

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "e"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "{\langle 1,g\rangle}"] \\ X \arrow[r, "{\langle 1,f\rangle}"'] & X \times Y \end{tikzcd}$$

is a pullback. The advantage of the latter formulation is that it consists entirely of monomorphisms. Hence it suffices to show that F preserves pullbacks of monomorphisms. Let the outer square below be such a pullback

$$\begin{tikzcd} L \arrow[rr, tail] \arrow[dd, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & & N \arrow[dl] \arrow[dd, tail, "n"] \\ & P \arrow[dr, dashed, "c"] & \\ M \arrow[ur] \arrow[rr, tail, "m"'] & & X, \end{tikzcd}$$

where P is the pushout, $P = M +_L N$. (Recall that, in a topos, the pushout of a monic is monic, so $M \rightarrowtail P$ and $N \rightarrowtail P$ are both monic.) Note that $L = M \cap_X N$. In fact, P is the union, $P = M \cup_X N$. This can be shown either categorically or, perhaps more easily, in the internal logic. (Internally speaking, P is the disjoint union of the two subsets M and N of X with two elements in the disjoint union being equal iff they come from the intersection of M and N. Thus P is the union of M and N.) Hence the canonical map $c\colon P \to X$ is monic.

By applying F to the inner diagram above we get a pushout of monics, since F preserves pushouts and monics. A pushout of monics is a pullback in a topos [Joh77]. Since the connecting map Fc is still monic, the outer square is then also a pullback. □

Proof of Lemma 7.3.35. We are to show that $LR\colon \mathcal{E} \to \mathcal{E}$ preserves products, monos, and all colimits. It preserves all colimits since both L and R are left adjoints.

To show that LR preserves the terminal object 1, it suffices to show that 1 is discrete. By Axiom 2, we can present 1 as a subquotient of a discrete object C,

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & C \\ 1. & \end{tikzcd}$$

Since $S \twoheadrightarrow 1$ is epic, it follows that the unique morphism from C to 1 is also epic. Hence 1 is a quotient of a discrete object, and thus discrete by Lemma 7.3.17.

Binary products are preserved by Axiom 4.

It remains to show that LR preserves monos. Thus let $m\colon M \rightarrowtail N$ be a monomorphism in $\mathcal{E}$. For clarity, let us denote the composite functor LR by d. We write $\epsilon\colon d \Rightarrow id$ for the counit of the adjunction $L \dashv R$. Consider the following diagram

$$\begin{tikzcd}
dM \arrow[dr, dotted, bend left=15, "u^\circ"] \arrow[ddrr, dashed, "u"] \arrow[ddrrr, bend left, "dm"] \arrow[dddrr, bend right, "\epsilon_M"'] & & & \\
& (m^*dN)^\circ \arrow[ul, dotted, bend left=15, "v"] \arrow[dr, tail, "a"] & & \\
& & m^*dN \arrow[r, tail, "c"] \arrow[d, "b"] \arrow[dr, phantom, "\lrcorner", very near start] & dN \arrow[d, "\epsilon_N"] \\
& & M \arrow[r, tail, "m"'] & N,
\end{tikzcd}$$

where the inner square is a pullback. The outer square commutes by definition of dm. Hence by the universal property of the pullback, there exists a unique morphism $u\colon dM \to m^*dN$ such that

$$bu = \epsilon_M \qquad \text{and} \qquad cu = dm.$$

Since $(m^*dN)^\circ$ is an open subobject of a discrete object dN, $(m^*dN)^\circ$ is discrete by Axiom 3. Hence by couniversality of ϵ_M, there exists a unique morphism $v\colon (m^*dN)^\circ \to dM$ such that

$$\epsilon_M v = ba.$$

We now claim that

$$vu^\circ = 1 \tag{7.10}$$
$$u^\circ v = 1 \tag{7.11}$$

that is, that dM is isomorphic to $(m^*dN)^\circ$, from which it easily follows that $dm = cau^\circ$ is monic, as required. Equation (7.10) follows from couniversality of ϵ_M ($\epsilon_M vu = bau^\circ = bu = \epsilon_M$, so $vu = 1$). To see that equation (7.11) holds, note that both for $x = a$ and for $x = au^\circ v$, we have that $mbx = \epsilon_N cx$. Therefore, by uniqueness of the mediating arrow to the pullback, we get that $a = au^\circ v$, from which it follows that $1 = u^\circ v$ since a is monic. □

Proof of Corollary 7.3.36. By Lemmas 7.3.35 and 7.3.34, the functor

$$LR\colon \mathcal{E} \to \mathcal{E}$$

is left exact. □

Proof of Lemma 7.3.37. We are to show that $L\colon \mathrm{D}_j\mathcal{E} \to \mathcal{E}$ is left exact.

L preserves finite products because the terminal object 1 in $\mathcal{E}$ discrete and also terminal in $\mathrm{D}_j\mathcal{E}$ and the product (formed in $\mathcal{E}$) of two discrete objects X and Y is again discrete by Axiom 4.

It remains to show that L preserves equalizers. We first show that L preserves monos.

So let

$$X \overset{m}{\rightarrowtail} Y$$

be a mono in $\mathrm{D}_j\mathcal{E}$. Apply functor L and form the image factorization of Lm in $\mathcal{E}$ to get

$$\begin{tikzcd} LX \arrow[rr, tail, "Lm"] \arrow[dr, two heads] & & LY \\ & I. \arrow[ur, tail] & \end{tikzcd}$$

Now apply functor R to get

$$\begin{tikzcd} X \arrow[rr, tail, "m"] \arrow[d, "\cong"'] & & Y \arrow[d, "\cong"] \\ RLX \arrow[rr, tail, "RLm"] \arrow[dr, two heads] & & RLY \\ & RI. \arrow[ur, tail] & \end{tikzcd}$$

Note that functor R preserves epis as a left adjoint and monos as right adjoint. Hence the morphism $RLX \to RI$ is epic. It is also monic (since by postcomposing with $RI \to RLY \cong Y$ we get a mono m), so iso. Thus $RLX \cong RI$.

Apply L again to get

$$\begin{tikzcd}
LX \arrow[rr, tail, "Lm"] \arrow[d, "\cong"'] & & LY \arrow[d, "\cong"] \\
LRLX \arrow[rr, tail, "LRLm"] \arrow[dr, "\cong"'] & & LRLY \\
& LRI. \arrow[ur, tail] &
\end{tikzcd}$$

Since LR is left exact by Corollary 7.3.36, LRI is the image factorization of Lm, so $LRI \rightarrowtail LRLY$ is monic, and hence Lm is monic. This completes the proof that L preserves monos.

We now proceed to show that L preserves equalizers. Let

$$\begin{tikzcd}
X \arrow[r, tail, "m"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z
\end{tikzcd}$$

be an equalizer in $\mathrm{D}_j\mathcal{E}$. Apply L and form the equalizer E of Lf and Lg in $\mathcal{E}$

$$\begin{tikzcd}
LX \arrow[r, tail, "Lm"] & LY \arrow[r, shift left, "Lf"] \arrow[r, shift right, "Lg"'] & LZ \\
E \arrow[ur, tail, "n"'] & &
\end{tikzcd}$$

Apply the functor R to get

$$\begin{tikzcd}
X \arrow[r, tail, "m"] \arrow[d, "\cong"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\cong"] & Z \arrow[d, "\cong"] \\
RLX \arrow[r, tail, "RLm"] & RLY \arrow[r, shift left, "RLf"] \arrow[r, shift right, "RLg"'] & RLZ \\
RE \arrow[ur, tail, "Rn"'] \arrow[uu, dashed, bend left=60, "u"] & &
\end{tikzcd}$$

Since $m\colon X \rightarrowtail Y$ is an equalizer, there exists a unique arrow $u\colon RE \to X$ such that $Rn = m \circ u$. Finally, apply L one more time to get

$$
\begin{tikzcd}
LX \arrow[r, tail, "Lm"] \arrow[d, "\cong"] \arrow[dd, dashed, bend right=60, "v"'] & LY \arrow[r, shift left, "Lf"] \arrow[r, shift right, "Lg"'] \arrow[d, "\cong"] & LZ \arrow[d, "\cong"] \\
LRLX \arrow[r, tail, "LRLm"] & LRLY \arrow[r, shift left, "LRLf"] \arrow[r, shift right, "LRLg"'] & LRLZ \\
LRE \arrow[uu, bend left=40, "Lu"] \arrow[ur, "LRn"'] & &
\end{tikzcd}
$$

Since LR is left exact by Corollary 7.3.36, LRn is the equalizer of Lf and Lg, so there is a unique arrow $v\colon LX \to LRE$, as shown in the diagram. It now suffices to show that

$$v \circ Lu = 1 \tag{7.12}$$

$$Lu \circ v = 1 \tag{7.13}$$

because then $LX \cong LRE$ and thus $Lm\colon LX \to LY$ is an equalizer. For equation (7.12), note that

$$LRn \circ v \circ Lu = Lm \circ Lu = LRn$$

from which we conclude that $v \circ Lu = 1$ since LRn is monic (since LR preserves monos by Corollary 7.3.36). For (7.13), note that

$$Lm \circ Lu \circ v = LRn \circ v = Lm$$

from which we get that $Lu \circ v = 1$, since Lm is monic (because m is and because L preserves monos, as shown above).

Thus L also preserves equalizers, and we have completed the proof of the lemma. □

We can now conclude that our axioms for localic local maps are *complete* in the sense that for a topos $\mathcal{E}$ with a topology j satisfying the axioms for localic local maps we indeed do get a localic local map from $\mathcal{E}$ to $\mathrm{D}_j\mathcal{E} \simeq \mathrm{Sh}_j\mathcal{E}$:

Corollary 7.3.38. *Let $\mathcal{E}$ be a topos with a topology j and suppose that $\mathcal{E}$ and j satisfy Axioms 1–4 for localic local maps. Then there is a localic local map from $\mathcal{E}$ to $\mathrm{D}_j\mathcal{E} \simeq \mathrm{Sh}_j\mathcal{E}$.*

Proof. By Theorems 7.3.31, 7.3.33, and 7.2.3 (see the explanation after Theorem 7.2.3). □

Soundness

The following proposition tells us that if we have a local map of toposes then the associated topology is principal.

Proposition 7.3.39. *Let $\mathcal{E}$ be a topos with topology j. If the associated sheaf functor $\mathbf{a}\colon \mathcal{E} \to \mathrm{Sh}_j\mathcal{E}$ has a left adjoint L, then the topology j is principal.*

Proof. By Lemma 7.3.5 it suffices to show that for every object X, there is a least dense subobject. Let ϵ denote the counit of the adjunction $L \dashv a$ and let $m\colon V \rightarrowtail X$ be an arbitrary dense subobject of X. Then by [Joh77, Proposition 3.42], $\mathbf{a}(m)\colon \mathbf{a}(V) \to \mathbf{a}(X)$ is an iso. Hence, also $L\,\mathbf{a}(m)\colon L\,\mathbf{a}(V) \to L\,\mathbf{a}(X)$ is an iso. Now, let U_X be the image of $\epsilon_X\colon L\,\mathbf{a}(X) \to X$ and let U_V be the image of $\epsilon_V\colon L\,\mathbf{a}(V) \to V$ and consider the following diagram

$$\begin{tikzcd}
& L\,\mathbf{a}\,V \arrow[rr, "\cong"] \arrow[dl, two heads] \arrow[dd, "\epsilon_V"] & & L\,\mathbf{a}\,X \arrow[dl, two heads] \arrow[dd, "\epsilon_X"] \\
U_V \arrow[dr, tail] & & U_X \arrow[ll, dashed, "\cong"] \arrow[dr, tail] & \\
& V \arrow[rr, tail, "m"'] & & X
\end{tikzcd}$$

Since $m\colon V \to X$ is monic, the image U_V of ϵ_V is the same as the image of $m \circ \epsilon_V = \epsilon_X \circ L\,\mathbf{a}(m)$ (the latter equality by naturality of ϵ). Since $L\,\mathbf{a}(m)$ is iso, the image of $\epsilon_X \circ L\,\mathbf{a}(m)$ is the image of ϵ_X, that is, U_X. In summa, U_V is isomorphic to U_X, as depicted in the diagram above. Hence $U_X \leq V$ for any dense subobject V. It remains to show that U_X is in fact dense. To this end, note that

$$\begin{tikzcd}
& \mathbf{a}\,L\,\mathbf{a}\,X \arrow[dl, two heads] \arrow[dd, "\mathbf{a}(\epsilon_X)"] \\
\mathbf{a}\,U_X \arrow[dr, tail] & \\
& \mathbf{a}\,X
\end{tikzcd}$$

is also an image factorization, as $\mathbf{a}$ preserves such (since $\mathbf{a}$ by assumption is both left and right adjoint). Moreover, by [MM92, Lemma 1, Section VII.4, Page 369], L is full and faithful since i is, so the unit $\eta\colon aL \Rightarrow id$ is iso.

Hence $\mathbf{a}(\epsilon_X)$ is iso (since $\mathbf{a}(\epsilon_X) \circ \eta_{\mathbf{a}X} = id$) and thus we also have that $\mathbf{a}\,U_X \rightarrowtail \mathbf{a}\,X$ is iso. We conclude by [Joh77, Proposition 3.42] that $U_X \rightarrowtail X$ is dense. $\square$

The following theorem expresses that our axioms are *sound.*

Theorem 7.3.40. *Every localic local map of toposes satisfies Axioms 1–4 for localic local maps.*

Proof. Let $\mathcal{E}$ and $\mathcal{F}$ be toposes with adjoint functors between them as in the situation

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\Gamma"] \\ \mathcal{F} \arrow[u, bend left=60, "\Delta"] \arrow[u, bend right=60, "\nabla"'] \end{tikzcd} \qquad \Delta \dashv \Gamma \dashv \nabla, \qquad \Delta \text{ full and faithful and lex.}$$

Suppose further that $\mathcal{E}$ is localic over $\mathcal{F}$, *i.e.*, that for all $X \in \mathcal{E}$, there exists a $C \in \mathcal{F}$ and a diagram of the form

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d] & \Delta C \\ X & \end{tikzcd}$$

in $\mathcal{E}$. Then by the discussion in Section 7.2, there is a topology j in $\mathcal{E}$ such that $\mathcal{F} \simeq \mathrm{Sh}_j\mathcal{E}$ and $\mathcal{F} \simeq \mathrm{D}_j\mathcal{E}$. So it suffices to show that Axioms 1–4 are satisfied. Axiom 1 holds by Proposition 7.3.39. Axiom 2 holds by the assumption that $\mathcal{E}$ is localic over $\mathcal{F}$, since the discrete objects are the replete image in $\mathcal{E}$ of $\mathcal{F}$ along Δ. Axiom 4 holds since Δ is lex. For Axiom 3 let $X \rightarrowtail C$ be an arbitrary subobject of a discrete object C. Consider the following diagram in $\mathcal{E}$

$$\begin{tikzcd} J(\widetilde{X^\circ}) \arrow[r, tail] \arrow[d, "e"] & C \cong J(\widetilde{C^\circ}) \arrow[dd, "\cong"] \\ X^\circ \arrow[d, tail] & \\ X \arrow[r, tail] & C, \end{tikzcd}$$

where J is the inclusion $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ of the discrete objects, and $X \mapsto \widetilde{X^\circ} \colon \mathcal{E} \to \mathrm{D}_j\mathcal{E}$ is the coreflector. Note that the top horizontal arrow in the diagram above is monic because the coreflector as a right adjoint preserves monos and because the inclusion of discrete objects is lex by the assumption that the given map is local. The vertical arrows are the counits of the coreflection adjunction, so the diagram commutes by naturality. Hence e is also monic and thus iso, so $\widetilde{X}^\circ \cong X^\circ$. Therefore, if X is open (*i.e.*, $X \cong X^\circ$), then $\widetilde{X}^\circ \cong X$ and X must be discrete. $\square$

7.3.2 Axioms for Bounded Local Maps

For $\mathcal{E}$ an elementary topos with a topology j we consider the following **axioms for bounded local maps**.

Axiom 1 j is principal.

Axiom 2a There is an object $G \in \mathcal{E}$ such that, for all $X \in \mathcal{E}$, there exists a discrete object C and a diagram

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d] & C \times G \\ X & \end{tikzcd}$$

in $\mathcal{E}$, presenting X as a subquotient of $C \times G$.

Axiom 2b There is a discrete object G' and a diagram

$$G' \twoheadrightarrow G^\circ \rightarrowtail G$$

in $\mathcal{E}$.

Axiom 3 For all discrete $C \in \mathcal{E}$, if $X \rightarrowtail C$ is open, then X is also discrete.

Axiom 4 For all discrete $C, C' \in \mathcal{E}$, $C \times C'$ is discrete.

Completeness

Theorem 7.3.41. *Let $\mathcal{E}$ be a topos with a topology j and suppose that $\mathcal{E}$ and j satisfy Axioms 1, 2a, 2b, 3 and 4 for bounded local maps. Then the category of discrete objects $\mathrm{D}_j\mathcal{E}$ is coreflective in $\mathcal{E}$.*

Proof. We show how to construct an associated discrete object for any object $X \in \mathcal{E}$. Let G, G', and S be as in Axioms 2a and 2b. The construction of the associated discrete object is contained in the following diagram:

$$\begin{tikzcd}
T^\circ \arrow[r, tail] \arrow[d, two heads] \arrow[ddd, two heads, bend right, "e"'] & T \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & C \times G' \arrow[d, two heads] \\
S^\circ \arrow[r, "\cong"] \arrow[dd, two heads] & S^\circ \arrow[r, tail, "f"] \arrow[d, tail] & C \times G^\circ \arrow[d, tail] \\
 & S \arrow[r, tail] \arrow[d, two heads] & C \times G \\
X^\circ \arrow[r, tail] & X. &
\end{tikzcd}$$

The right-most vertical arrows are induced in the obvious way from the arrows $G' \twoheadrightarrow G^\circ \rightarrowtail G$ given by Axiom 2b. The morphism $f\colon S^\circ \to C \times G^\circ$ is obtained as follows. Applying the interior functor to the morphism $S \rightarrowtail C \times G$ gives a morphism $S^\circ \rightarrowtail (C \times G)^\circ$, monic by Lemma 7.3.7. Moreover, applying the interior functor to the product projections $C \xleftarrow{\pi} C \times G \xrightarrow{\pi'} G$ gives morphisms

$$C^\circ \cong C \longleftarrow (C \times G)^\circ \longrightarrow G^\circ,$$

where $C^\circ \cong C$ since C is discrete and thus open by Lemma 7.3.15. It follows that there is a unique morphism from $(C \times G)^\circ$ to the product $C \times G^\circ$. Composing this morphism with the $S^\circ \rightarrowtail (C \times G)^\circ$ gives the morphism f shown in the diagram. It can be verified that f then is monic. Object T is obtained as a pullback as shown. Since pulling back in a topos preserves epimorphisms, $T \twoheadrightarrow S^\circ$ is epic. The morphism $T^\circ \twoheadrightarrow S^\circ$ is the interior functor applied to the epi $T \twoheadrightarrow S^\circ$; it is epic since the interior functor preserves epis by Lemma 7.3.11. By Axiom 4, $C \times G'$ is discrete and thus, by Axiom 3, also T° is discrete. Finally, the epi $S^\circ \twoheadrightarrow X^\circ$ is obtained by applying the interior functor to $S \twoheadrightarrow X$, and e is the composite epi $T^\circ \twoheadrightarrow S^\circ \twoheadrightarrow X^\circ$.

Now take the exterior of e to get $\widetilde{X^\circ}$. By the same argument as in the proof of Theorem 7.3.31, this is the associated discrete object of X. □

Theorem 7.3.42. *Let $\mathcal{E}$ be a topos with a topology j and suppose that $\mathcal{E}$ and j satisfy Axioms 1–4 for bounded local maps. Then the inclusion $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ is lex.*

Proof. As the proof of Theorem 7.3.33. □

Corollary 7.3.43. *Let $\mathcal{E}$ be a topos with a topology j and suppose that $\mathcal{E}$ and j satisfy Axioms 1–4 for bounded local maps. Then there is a bounded local map from $\mathcal{E}$ to $\mathrm{D}_j\mathcal{E} \simeq \mathrm{Sh}_j\mathcal{E}$.*

Proof. By Theorems 7.3.41, 7.3.42, and 7.2.3 (see the explanation after Theorem 7.2.3). □

Soundness

Theorem 7.3.44. *Every bounded local map of toposes satisfies the axioms for bounded local maps.*

Proof. The proof proceeds as the proof of Theorem 7.3.40. With notation as in that proof, Axiom 2b holds since the associated sheaf functor $\mathbf{a}$ has a left adjoint L and we have, by Proposition 7.3.39, a diagram of the form

$$LaG \twoheadrightarrow G^\circ \rightarrowtail G$$

in $\mathcal{E}$. Since $\mathrm{Sh}_j\mathcal{E} \simeq \mathrm{D}_j\mathcal{E}$, we have that LaG is isomorphic to a discrete object, proving that Axiom 2b holds. Axiom 2a holds by the assumption that the given local map is bounded. Axioms 1, 3, and 4 hold as in the proof of Theorem 7.3.40. □

7.4 More Properties of Open Objects

Convention 7.4.1. For the remainder of this section we assume given a bounded local map $f\colon \mathcal{E} \to \mathcal{F}$ of toposes. Then there is a principal topology j in $\mathcal{E}$ such that $\mathcal{F} \simeq \mathrm{Sh}_j\mathcal{E}$ and the associated sheaf functor $\mathbf{a}$ has a left adjoint L.

Under the assumptions stated in the above convention we now draw some easy conclusions about the open objects.

Corollary 7.4.2. *Any open object in $\mathcal{E}$ is a quotient of a discrete object.*

Proof. By Corollary 7.3.32 and the proof of Proposition 7.3.39, the interior X° of an object X is obtained as the image of the counit $\epsilon_X : L\,\mathbf{a}\,X \to X$ of $L \dashv \mathbf{a}$:

$$L\,\mathbf{a}\,X \twoheadrightarrow X^\circ \rightarrowtail X.$$

Thus, if X is open (*i.e.*, $X^\circ \cong X$), the image of ϵ_X is X and X is a quotient of $L\,\mathbf{a}\,X$. □

Remark 7.4.3. Combining the above corollary with Lemma 7.3.15 and Lemma 7.3.17 we conclude that: *The open objects of $\mathcal{E}$ are exactly the quotients of the discrete objects in $\mathcal{E}$.*

Proposition 7.4.4. *The category of open objects $\mathrm{O}_j\mathcal{E}$ is closed under finite colimits in $\mathcal{E}$.*

Proof. The initial object is discrete, hence open. By Corollary 7.4.2, any two open objects X and Y are both quotients of discrete objects, say C_X and C_Y. Thus also the coproduct $X+Y$ is a quotient of the discrete object C_X+C_Y (C_X+C_Y is discrete by Lemma 7.3.2) and hence $X+Y$ is open by Lemma 7.3.17. Finally, a coequalizer of a pair of morphisms between two open objects, each covered by a discrete object, is of course also covered by a discrete object, hence is open by Lemma 7.3.17. $\square$

Proposition 7.4.5. *The interior functor $X \mapsto X^\circ \colon \mathcal{E} \to \mathcal{E}$ preserves preserves finite products.*

Proof. To show that the interior functor preserves the terminal object 1, it suffices to show that 1 is open. But we already know that 1 is discrete and hence it is also open.

Now for binary products, let X, Y be objects in $\mathcal{E}$. Consider the following diagram in $\mathcal{E}$

$$\begin{tikzcd}
L\mathbf{a}X \arrow[d, "e_1"'] & L\mathbf{a}X \times L\mathbf{a}Y \arrow[l] \arrow[r] \arrow[d, "e"'] & L\mathbf{a}Y \arrow[d, "e_2"] \\
X^\circ \arrow[d, tail, "m_1"'] & X^\circ \times Y^\circ \arrow[l] \arrow[r] \arrow[d, tail, "m"'] & Y^\circ \arrow[d, tail, "m_2"] \\
X & X \times Y \arrow[l] \arrow[r] & Y,
\end{tikzcd}$$

where $m_i e_i$ are the image factorizations of the counits ϵ_X and ϵ_Y of the coreflection $L \dashv \mathbf{a}$. The morphism e is $e_1 \times e_2$ and the morphism m is $m_1 \times m_2$, so e is epic and m is monic. Now since $\mathbf{a}$ preserves products, $\mathbf{a}(X \times Y) \cong \mathbf{a}X \times \mathbf{a}Y$. By Axiom 4, L preserves binary products, so $L\mathbf{a}(X \times Y) \cong L\mathbf{a}X \times L\mathbf{a}Y$. Thus, $(X \times Y)^\circ$, the image factorization of the counit $\epsilon_{X\times Y}$, is isomorphic to the image factorization of $\epsilon_X \times \epsilon_Y = me$, that is, $(X \times Y)^\circ \cong X^\circ \times Y^\circ$, as required. $\square$

Corollary 7.4.6. *The category $\mathrm{O}_j\mathcal{E}$ of open objects has finite products and they are computed as in $\mathcal{E}$ and thus preserved by the inclusion $\mathrm{O}_j\mathcal{E} \hookrightarrow \mathcal{E}$.*

Proof. The terminal object is open. Let L denote the inclusion functor $\mathrm{O}_j\mathcal{E} \hookrightarrow \mathcal{E}$ and let R denote the right adjoint $X \mapsto X^\circ$, see Lemma 7.3.20. From the adjunction $L \dashv R$ is follows that the product $X \times_{\mathrm{O}_j\mathcal{E}} Y$ of $X, Y \in \mathrm{O}_j\mathcal{E}$ in $\mathrm{O}_j\mathcal{E}$ is $R(LX \times_{\mathcal{E}} LY)$, where $\times_{\mathcal{E}}$ denotes the product in $\mathcal{E}$. But $R(LX \times_{\mathcal{E}} LY) = (X \times_{\mathcal{E}} Y)^\circ \cong X^\circ \times Y^\circ \cong X \times_{\mathcal{E}} Y$ by Proposition 7.4.5 and since X and Y are open. □

Proposition 7.4.7. *Let $f\colon C \to D$ be a morphism between discrete objects C and D in $\mathcal{E}$ and suppose $X \rightarrowtail D$ is an open subobject of D. Then $f^*(X)$, the pullback of X along f, as in the diagram*

$$\begin{tikzcd}
f^*(X) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail] \\
C \arrow[r, "f"'] & D,
\end{tikzcd}$$

is open.

Proof. By Axiom 3, X is discrete (as an open subobject as a discrete object). The discrete objects are closed under finite limits in $\mathcal{E}$ by Theorem 7.3.42, and thus $f^*(X)$ is also discrete, and hence open by Lemma 7.3.15. □

Chapter 8

Logic and Local Maps of Toposes

Suppose given a topos $\mathcal{E}$ with a topology j satisfying the axioms for bounded local maps set out in the previous chapter. There results a local map of toposes

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\Gamma"] \\ \mathrm{D}_j\mathcal{E} \arrow[u, bend left=60, "\Delta"] \arrow[u, bend right=60, "\nabla"'] \end{tikzcd} \qquad \Delta \dashv \Gamma \dashv \nabla$$

with Δ the inclusion of the discrete objects and Γ the associated discrete object functor with right adjoint ∇. In this chapter we ask: What can we then say about the relationship between the internal logic of the topos of discrete objects $\mathrm{D}_j\mathcal{E}$ and the internal logic of $\mathcal{E}$ itself ? In particular, we would like to know when the interpretation of a sentence with basic types and predicates interpreted in $\mathrm{D}_j\mathcal{E}$ agrees with the interpretation of the sentence in $\mathcal{E}$. This way we can obtain information about $\mathcal{E}$ in terms of $\mathrm{D}_j\mathcal{E}$. This approach is advantageous in situations where we have a better understanding of the topos of discrete objects than of the topos $\mathcal{E}$ itself, for example, in the case where $\mathcal{E}$ is the relative realizability topos $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{D}_j\mathcal{E}$ is the standard realizability topos $\mathrm{RT}(A_\sharp)$. Our point of view is thus analogous to the point of view of Hyland who investigates when the logic of a sheaf subtopos (**Set**) agrees with the logic of a given topos (the effective topos) [Hyl82, Section 5].

In Section 8.1 we briefly recall the logic of sheaves and its relation to the logic of $\mathcal{E}$. In Section 8.2 we then develop the logic of discrete objects.

In more details, we define a fibration of open subobjects, prove that it is equivalent to the fibration of closed subobjects, and that the logic of $\mathrm{D}_j\mathcal{E}$ is obtained from the fibration of open subobjects by change-of-base along the inclusion of discrete objects. This is much like the logic of sheaves, which is obtained from the fibration of closed subobject by change-of-base along the inclusion of the sheaves. Thus we extend the adjoint-cylinder picture to the logics of sheaves and discrete objects. We prove that the interpretation of a certain class of stable formulas (encompassing geometric formulas, of course) is preserved by the inclusion of the discrete objects into $\mathcal{E}$. In Section 8.3 we define a modal logic for local maps. We both describe the syntactic calculus and also its interpretation given any local map of toposes. The modality $\sharp$ is interpreted by the interior operator and it satisfies the usual properties of the box operator from S4. The modal logic can be seen as a kind of internal logic for local maps (resp. local toposes over a fixed base topos) and it is useful to obtain more relationships between the logic of discrete objects and the logic of $\mathcal{E}$. We give two sample applications in this direction.

Much as the closure operation is important when relating the logic of sheaves to the logic of $\mathcal{E}$, the interior operation is important, when relating the logic of the discrete objects to the logic of $\mathcal{E}$. The closure operation is a *logical operation* in the logic of $\mathcal{E}$, in the sense that it is a map on the subobject classifier Ω of $\mathcal{E}$ or, equivalently, it is a natural operation on subobjects which commutes with pullback. In logical terms, this means that the closure operation commutes with substitution (as one would expect of any well-behaved logical connective / operation). For the interior operation this is *not* the case, see Remark 7.3.4. This fact has two consequences: (1) when we describe the fibration of open subobjects in Section 8.2, substitution (reindexing) is defined in a slightly more subtle way than usual; and (2) when we consider the interior operator as a modal operator in our modal logic for local maps in Section 8.3, we restrict attention to a subcollection of types from $\mathcal{E}$ satisfying that for predicates on these types interior does indeed commute with substitution. We remark that it is classical that there is a problem of noncommutativity of substitution with respect to modal operators, see [GM87] and the references therein.

8.1 The Logic of Sheaves

In this section we briefly recall how the logic of sheaves, *i.e.*, the internal logic of the subobject fibration on $\mathrm{Sh}_j\mathcal{E}$, relate to the logic of $\mathcal{E}$. The material presented in this section is standard; we follow [Jac99], see also [Hyl82, MM92].

Proposition 8.1.1. *Let $j\colon \Omega \to \Omega$ be a Lawvere-Tierney topology in a topos $\mathcal{E}$ and let Ω_j be the image of j. Since closure commutes with pullback, we get a split fibration* $\begin{smallmatrix}\mathrm{ClSub}_j(\mathcal{E})\\ \downarrow\\ \mathcal{E}\end{smallmatrix}$ *of closed subobjects. It is a higher-order fibration with extensional entailment, in which:*

- $\top_j$, $\wedge_j$, $\supset_j$, *and* $\forall_j$ *are as for ordinary subobjects.*
- $\bot_j = \overline{\bot}$, $X \vee_j Y = \overline{X \vee Y}$, $\exists_j(X) = \overline{\exists(X)}$, *and* $\mathrm{Eq}_j(X) = \overline{\mathrm{Eq}(X)}$, *and thus* $\neg_j(X) = X \supset \overline{\bot}$.
- **true**: $1 \rightarrowtail \Omega_j$ *is a split generic object.*

We have here labelled the connectives etc. *in* $\begin{smallmatrix}\mathrm{ClSub}_j(\mathcal{E})\\ \downarrow\\ \mathcal{E}\end{smallmatrix}$ *with a subscript j. Hence closure $\overline{(-)}$ defines a fibred functor* $\mathrm{Sub}(\mathcal{E}) \to \mathrm{ClSub}_j(\mathcal{E})$ *over $\mathcal{E}$ which preserves all this structure except the generic object.*

For the proof of the proposition, see, *e.g.*, [Jac99, Proposition 5.6.6]. Suffice it here to recall that closure commutes with finite meets and that for subobjects $X, Y \in \mathrm{Sub}(Y)$,

$$X \supset \overline{Y} = \overline{X} \supset \overline{Y} = \overline{\overline{X} \supset \overline{Y}}$$

(Both equalities are not too difficult to show; the first can be found explicitly in [Wil94, Proposition 9.11, Page 46], the second can be found in [Jac99, Proposition 5.6.6]. One can also see these equalities as an application of Freyd's theorem [FS90] that a full reflective subcategory is an exponential ideal if the reflector (in this case closure) preserves products.) Also, for $X \in \mathrm{ClSub}_j(\mathcal{E})(I \times J)$, *i.e.*, X closed,

$$\forall j\colon J.\ X = \overline{\forall j\colon J.\ X}.$$

Let us write out explicitly what the above proposition says with regard to first-order logic. For all objects (types) $I, J \in \mathcal{E}$, all closed subobjects (predicates in the closed subobject fibration) $X, Y \in \mathrm{ClSub}_j(I)$ and $Z \in$

$\mathrm{ClSub}_j(I \times J)$, and morphisms (terms) $x, x' \colon 1 \to X$

$$
\begin{aligned}
(x =_j x') &= \overline{x = x'} \\
\top_j &= \top \\
X \wedge_j Y &= X \wedge Y \\
\bot_j &= \overline{\bot} \\
X \vee_j Y &= \overline{X \vee Y} \\
X \supset_j Y &= X \supset Y \\
\neg_j X &= X \supset \overline{\bot} \\
\exists_j j \colon J.\ Z &= \overline{\exists j \colon J.\ Z} \\
\forall_j j \colon J.\ Z &= \forall j \colon J.\ Z
\end{aligned}
$$

Moreover, substitution in $\begin{smallmatrix}\mathrm{ClSub}_j(\mathcal{E}) \\ \downarrow \\ \mathcal{E}\end{smallmatrix}$ is interpreted as it is in $\begin{smallmatrix}\mathrm{Sub}(\mathcal{E}) \\ \downarrow \\ \mathcal{E}\end{smallmatrix}$, *i.e.*, by pullback.

Recall that when $I \in \mathcal{E}$ is a sheaf, a subobject $X \rightarrowtail I$ in $\mathcal{E}$ is closed iff $X \rightarrowtail I$ is a subobject in $\mathrm{Sh}_j\mathcal{E}$. Therefore one can show the following proposition (see, *e.g.*, [Jac99, 5.7.11] and [Joh77] for more details).

Proposition 8.1.2. *There is a change-of-base situation*

$$
\begin{tikzcd}
\mathrm{Sub}(\mathrm{Sh}_j\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{ClSub}_j(\mathcal{E}) \arrow[d] \\
\mathrm{Sh}_j\mathcal{E} \arrow[r, hook] & \mathcal{E}.
\end{tikzcd}
$$

Propositions 8.1.1 and 8.1.2 taken together give us a *translation* of the first-order logic of $\mathrm{Sh}_j\mathcal{E}$ into the first-order logic of $\mathcal{E}$. For φ first-order formula with basic types and basic predicates interpreted as sheaves, write $[\![\varphi]\!]_{\mathrm{Sh}_j\mathcal{E}}$ for the interpretation of φ in the subobject fibration on $\mathrm{Sh}_j\mathcal{E}$. If we view the interpretation of the basic types as objects of $\mathcal{E}$ and the interpretation of the atomic predicates as closed subobjects on those objects in $\mathcal{E}$, the interpretation $[\![\varphi]\!]_j$ of φ in the closed subobject fibration equals $[\![\varphi]\!]_{\mathrm{Sh}_j\mathcal{E}}$. Moreover, by Proposition 8.1.1, if φ is built up from atomic predicates and $\top$, $\wedge$, $\supset$, and $\forall$ (*i.e.*, φ is a **negative formula**), then $[\![\varphi]\!]_j$ equals $[\![\varphi]\!]$, the interpretation of φ in logic of $\mathcal{E}$. For more results of this nature, see [Hyl82].

8.2 The Logic of Discrete Objects

Recall from Chapter 7 that when $I \in \mathcal{E}$ is a discrete object, a subobject $X \rightarrowtail I$ is open iff X is discrete iff $X \rightarrowtail I$ is a subobject in $\mathrm{D}_j\mathcal{E}$. We will use this fact to get results, analogous to those for sheaves recalled in the previous section, relating the logic of the discrete objects to the logic of $\mathcal{E}$.

Definition 8.2.1. We define $\mathrm{OpenSub}_j(\mathcal{E})$ to be the full subcategory of $\mathrm{Sub}(\mathcal{E})$ on the open subobjects.

Proposition 8.2.2. *The codomain functor* $\mathrm{cod}\colon \mathrm{OpenSub}_j(\mathcal{E}) \to \mathcal{E}$ *is a fibration* $\begin{array}{c}\mathrm{OpenSub}_j(\mathcal{E})\\ \downarrow \\ \mathcal{E}\end{array}$ *with reindexing of* $X \rightarrowtail J$ *along* $u\colon I \to J$ *given by* $u^*(X)^\circ$*, the interior of the pullback of* X *along* u*.*

Proof. Consider the following diagrams.

$$
\begin{tikzcd}
Y \arrow[r, dashed, "h"] \arrow[d, tail, "\psi"'] \arrow[dr, dashed, "h'"'] \arrow[drr, "g"] & I \times_J X^\circ \arrow[d, tail, "m"] \arrow[dr, "\overline{u}"] & \\
H \arrow[dr, dashed, "w"'] \arrow[drr, "v"] & I \times_J X \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "\varphi"] \\
& I \arrow[r, "u"'] & J
\end{tikzcd}
$$

$$
\begin{tikzcd}
H \arrow[dr, dashed, "w"'] \arrow[drr, "v"] & & \\
& I \arrow[r, "u"'] & J
\end{tikzcd}
$$

Here $\varphi\colon X \rightarrowtail J$ is an open subobject over J and $u\colon I \to J$ is a map in the base category $\mathcal{E}$. $I \times_J X^\circ$ is the interior of the pullback of X along u and $\overline{u}\colon I \times_J X^\circ \to X$ is obtained as the interior functor applied to the morphism $I \times_J X \to X$ (here we use that X is open, *i.e.*, $X \cong X^\circ$). We claim that $\overline{u}$ is a cartesian lifting over u. Thus suppose that $\psi\colon Y \rightarrowtail H$ is an open subobject of H and that (g, v) is a map from $\psi\colon Y \rightarrowtail H$ to $\varphi\colon X \rightarrowtail J$ in $\mathrm{OpenSub}_j(\mathcal{E})$, *i.e.*, $\varphi g = v\psi$, see the diagram above. Suppose further that v factors as uw, for some $w\colon H \to I$, shown in the diagram. Then by the universal property of the pullback, there exists an $h'\colon Y \to I \times_J X$ such that $w\psi$ and φm both factor through h'. Let h be the interior of h'. Then

$$\overline{u}h = g \qquad \text{and} \qquad uw = v$$

so $(g, v)\colon \psi \to \varphi$ in $\mathrm{OpenSub}_j(\mathcal{E})$ via factors via (h, w). Moreover, (h, w) is the unique such over w since m is monic, completing the proof that $\overline{u}$ is cartesian over u and thus that $\begin{array}{c}\mathrm{OpenSub}_j(\mathcal{E})\\ \downarrow\\ \mathcal{E}\end{array}$ is a fibration. □

Proposition 8.2.3. *The interior operation and the closure operation establish a fibred equivalence as in*

$$\begin{tikzcd} \mathrm{OpenSub}_j(\mathcal{E}) \arrow[rr, bend left, "\overline{(\ )}"] \arrow[rr, phantom, "\simeq"] \arrow[dr] & & \mathrm{ClSub}_j(\mathcal{E}) \arrow[ll, bend left, "\circ"] \arrow[dl] \\ & \mathcal{E}. & \end{tikzcd}$$

Proof. Note first that by Corollary 7.3.8,

1. for $X \rightarrowtail I$ an open subobject, $\overline{X}^{\circ} = X^{\circ} = X$
2. for $X \rightarrowtail I$ a closed subobject, $\overline{X^{\circ}} = \overline{X} = X$

Thus it only remains to show that closure and interior are *fibred* functors. Closure is fibred because, for any open $X \rightarrowtail J$ and any map $u\colon I \to J$ in the base, $\overline{(u^*X^{\circ})} = \overline{(u^*X)} = u^*(\overline{X})$. Interior is fibred because, for any closed $X \rightarrowtail J$ and any map $u\colon I \to J$ in the base, $(u^*X)^{\circ} = (u^*(X^{\circ}))^{\circ}$ (the $\geq$ is obvious, for the other direction use adjointness: $(u^*X)^{\circ} \leq (u^*(X^{\circ}))^{\circ}$ iff $u^*X \leq \overline{u^*(X^{\circ})}$ iff $u^*X \leq u^*\overline{X}$, which holds). □

Proposition 8.2.4. *The fibration* $\begin{array}{c}\mathrm{OpenSub}_j(\mathcal{E})\\ \downarrow\\ \mathcal{E}\end{array}$ *of open subobjects is a higher-order fibration with extensional entailment, in which: (we label the connective* etc. *in* $\begin{array}{c}\mathrm{OpenSub}_j(\mathcal{E})\\ \downarrow\\ \mathcal{E}\end{array}$ *with a subscript* $\circ$*)*

- $\bot_{\circ}$, $\vee_{\circ}$, $\exists_{\circ}$, $\mathrm{Eq}_{\circ}$ *are as for ordinary subobjects.*
- $\top_{\circ} = \top^{\circ}$, $X \wedge_{\circ} Y = (X \wedge Y)^{\circ}$, $X \supset_{\circ} Y = (X \supset \overline{Y})^{\circ}$, $(\forall_{\circ})_f X = (\forall_f \overline{X})^{\circ}$, *and thus* $\neg_{\circ}(X) = (X \supset \overline{\bot})^{\circ}$.
- **true**: $1 \rightarrowtail \Omega$ *is a split generic object.*

Hence interior $(-)^{\circ}$ *defines a fibred functor* $\mathrm{Sub}(\mathcal{E}) \to \mathrm{OpenSub}_j(\mathcal{E})$ *over* $\mathcal{E}$ *which preserves all this structure, except the generic object.*

Proof. The first-order structure is defined categorically and thus preserved along equivalences. Hence we can use Proposition 8.2.3 to derive what the logical operations are in $\begin{matrix}\mathrm{OpenSub}_j(\mathcal{E})\\ \downarrow\\ \mathcal{E}\end{matrix}$: Let I and J be objects of $\mathcal{E}$, $f\colon I \to J$ in $\mathcal{E}$, and suppose X, Y are open subobjects of J. Then we have:

- $\bot_{\mathrm{o}} = {\bot_j}^{\circ} = \overline{\bot}^{\circ} = \bot^{\circ} = \bot$ since $\bot$ is the initial object which is discrete and thus open.
- For $\vee_{\mathrm{o}}$ we argue as follows.

$$\begin{aligned} X \vee_{\mathrm{o}} Y &= \overline{X} \vee_j \overline{Y}^{\circ} \\ &= \overline{\overline{X} \vee \overline{Y}}^{\circ} \\ &= (\overline{X} \vee \overline{Y})^{\circ} \\ &= X \vee Y^{\circ} \quad \text{see below} \\ &= X \vee Y \quad \text{by Proposition 7.4.4} \end{aligned}$$

  For the equation $(\overline{X} \vee \overline{Y})^{\circ} = X \vee Y^{\circ}$ note that we clearly have that the $\geq$ direction holds. For the other direction note that by adjointness

$$(\overline{X} \vee \overline{Y})^{\circ} \leq X \vee Y^{\circ}$$

  holds iff

$$(\overline{X} \vee \overline{Y}) \leq \overline{X \vee Y} \tag{8.1}$$

  holds. But clearly $\overline{X} \leq \overline{X \vee Y}$ and likewise $\overline{Y} \leq \overline{X \vee Y}$, and therefore (since $\vee$ is the least upper bound operation) we have that (8.1) holds.

- For $\exists_{\mathrm{o}}$ we argue as follows.

$$\begin{aligned} (\exists_{\mathrm{o}})_f X &= (\exists_j)_f \overline{X}^{\circ} \\ &= \overline{\exists_f \overline{X}}^{\circ} \\ &= (\exists_f \overline{X})^{\circ} \\ &= \exists_f \overline{X}^{\circ} \quad \text{by Lemma 7.3.9} \\ &= \exists_f X^{\circ} \\ &= \exists_f X \quad \text{since } X \text{ is open} \end{aligned}$$

- $\mathrm{Eq}_\circ$ is a special case of $\exists_\circ$ (since we show that we have left adjoints $\exists_f$ for *all* morphisms f in the base, not only the projections).

- $\top_\circ = \top_j{}^\circ = \top^\circ$.

- For $\wedge_\circ$ we argue as follows:

$$\begin{aligned} X \wedge_\circ Y &= (\overline{X} \wedge_j \overline{Y})^\circ \\ &= (\overline{X} \wedge \overline{Y})^\circ \\ &= \overline{(X \wedge Y)}^\circ \\ &= (X \wedge Y)^\circ \end{aligned}$$

- For $\supset_\circ$ we argue as follows:

$$\begin{aligned} X \supset_\circ Y &= (\overline{X} \supset_j \overline{Y})^\circ \\ &= (\overline{X} \supset \overline{Y})^\circ \\ &= (X \wedge \overline{Y})^\circ. \end{aligned}$$

- For $\forall_\circ$ we argue as follows:

$$\begin{aligned} (\forall_\circ)_f X &= ((\forall_j)_f \overline{X})^\circ \\ &= (\forall_f \overline{X})^\circ. \end{aligned}$$

It only remains to show that **true**: $1 \rightarrowtail \Omega$ is a split generic object. Let $X \rightarrowtail I$ be an open subobject of I and let $\chi\colon I \to \Omega$ be the characteristic map of X in $\mathcal{E}$, *i.e.*, χ is the unique map making the square below a pullback diagram.

$$\begin{tikzcd} X^\circ \cong X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathbf{true}"] \\ I \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

Since 1 is open, the interior of X is isomorphic to X. Thus the reindexing of **true**: $1 \to \Omega$ along χ in $\begin{array}{c}\mathsf{OpenSub}_j(\mathcal{E})\\ \downarrow \\ \mathcal{E}\end{array}$, that is, the interior of the pullback of 1 along χ, is X itself. $\square$

Let us write out explicitly what the above proposition says with regard to first-order logic. For all objects (types) $I, J \in \mathcal{E}$, all open subobjects (predicates in the open subobject fibration) $X, Y \in \mathrm{OpenSub}_j(I)$ and $Z \in \mathrm{OpenSub}_j(I \times J)$, and morphisms (terms) $x, x' \colon 1 \to X$

$$\begin{aligned}
(x =_\circ x') &= (x = x') \\
\top_\circ &= \top^\circ \\
X \wedge_\circ Y &= (X \wedge Y)^\circ \\
\bot_\circ &= \bot \\
X \vee_\circ Y &= X \vee Y \\
X \supset_\circ Y &= (X \supset \overline{Y})^\circ \\
\neg_\circ X &= (X \supset \overline{\bot})^\circ \\
\exists_\circ j \colon J.\ Z &= \exists j \colon J.\ Z \\
\forall_\circ j \colon J.\ Z &= (\forall j \colon J.\ \overline{Z})^\circ
\end{aligned}$$

Moreover, substitution in $\begin{smallmatrix}\mathrm{OpenSub}_j(\mathcal{E})\\ \downarrow \\ \mathcal{E}\end{smallmatrix}$ is interpreted as the interior of the pullback.

Proposition 8.2.5. *There is a change-of-base situation*

$$\begin{tikzcd}
\mathrm{Sub}(\mathrm{D}_j\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{OpenSub}_j(\mathcal{E}) \arrow[d] \\
\mathrm{D}_j\mathcal{E} \arrow[r, hook] & \mathcal{E}.
\end{tikzcd}$$

Proof. For $X \rightarrowtail J$ an open subobject of a discrete object J. Then X itself is discrete, by Axiom 3 in Section 7.3.2. Moreover, since the discrete objects are closed under finite limits in $\mathcal{E}$, the pullback $u^*(X)$ of X along a map $u \colon I \to J$ between discrete objects is discrete and hence also open. Thus the reindexing of X along u in $\begin{smallmatrix}\mathrm{OpenSub}_j(\mathcal{E})\\ \downarrow \\ \mathcal{E}\end{smallmatrix}$, namely $u^*(X)^\circ$ is equal to (as subobjects of I) the reindexing of X in $\mathrm{Sub}(\mathrm{D}_j\mathcal{E})$, namely $u^*(X)$. □

Combining the above proposition with Proposition 8.1.2 we have the following picture, complementing the adjoint cylinder picture (where the discrete objects come in to $\mathcal{E}$ on the left, the sheaves come in to $\mathcal{E}$ on the right, and the category of discrete objects is equivalent to the category of sheaves).

$$
\begin{tikzcd}
\mathrm{Sub}(\mathrm{D}_j\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{OpenSub}_j(\mathcal{E}) \arrow[rr, bend left, "\overline{}"] \arrow[rr, phantom, "\simeq"] \arrow[dr] & & \mathrm{ClSub}_j(\mathcal{E}) \arrow[ll, bend left, "\circ"] \arrow[dl] & \mathrm{Sub}(\mathrm{Sh}_j\mathcal{E}) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathrm{D}_j\mathcal{E} \arrow[rr, hook] & & \mathcal{E} & & \mathrm{Sh}_j\mathcal{E} \arrow[ll, hook']
\end{tikzcd}
$$

Combining Propositions 8.2.5 and 8.2.4, we of course derive a translation of the internal logic of $\mathrm{D}_j\mathcal{E}$ into the logic of $\mathcal{E}$. Since we are restricting attention to the discrete objects in the base, we can make some simplifications compared to what we get directly from Proposition 8.2.4 (see also the explicit treatment after that proposition): Since open subobjects X and Y of a *discrete* object I are in fact discrete (by Axiom 3) and since the inclusion of discrete objects is left exact, $X \wedge_\circ Y = (X \wedge Y)^\circ$ simplifies to $X \wedge_\circ Y = X \wedge Y$ and $\top_\circ = \top^\circ$ simplifies to $\top$. Moreover, we have the following two lemmas, which tell us that we can simplify the definition of $\supset$ and $\forall$.

Lemma 8.2.6. *Let I be an object of $\mathcal{E}$ and let $X, Y \in \mathrm{Sub}_{\mathcal{E}}(I)$ be subobjects of I. Suppose that I is discrete and that X is open. Then $(X \supset Y)^\circ = (X \supset Y^\circ)^\circ$.*

Proof. Note first that $(X \supset Y)^\circ \leq X \supset Y^\circ$:

$$
\begin{array}{clr}
 & (X \supset Y)^\circ \leq X \supset Y^\circ & \\
\iff & (X \supset Y)^\circ \wedge X \leq Y^\circ & \\
\iff & (X \supset Y)^\circ \wedge X^\circ \leq Y^\circ & \text{since } X \text{ is open} \\
\iff & (X \supset Y \wedge X)^\circ \leq Y^\circ & \text{since } (X \supset Y)^\circ,\ X^\circ \text{ both disc.} \\
\iff & Y^\circ \leq Y^\circ &
\end{array}
$$

Hence, using that interior is idempotent, we also have that $(X \supset Y)^\circ \leq X \supset Y^{\circ\circ}$. The other direction, $(X \supset Y^\circ)^\circ \leq (X \supset Y)^\circ$ is obvious, since $Y^\circ \leq Y$. $\square$

Lemma 8.2.7. *Let $u \colon I \to J$ be a morphism of discrete objects in $\mathcal{E}$ and let $X \in \mathrm{Sub}_{\mathcal{E}}(I)$ be a subobject of I. Then $(\forall_u \overline{X})^\circ = (\forall_u X)^\circ$.*

Proof. It is clear that $(\forall_u X)^\circ \leq (\forall_u \overline{X})^\circ$ since $X \leq \overline{X}$. For the other direction we reason as follows. By adjointness, $(\forall_u \overline{X})^\circ \leq (\forall_u X)^\circ$ holds iff

$$\forall_u \overline{X} \leq \overline{(\forall_u X)^\circ} \leq \overline{\forall_u X}.$$

We show that in fact

$$\forall_u \overline{X} = \overline{\forall_u X}. \tag{8.2}$$

By taking left adjoints (8.2) holds iff, for any subobject Y of J,

$$u^* Y^\circ = (u^* Y)^\circ.$$

But, by Proposition 7.4.7, $u^*Y^\circ = (u^*Y^\circ)^\circ$ and clearly $(u^*Y^\circ)^\circ \leq (u^*Y)^\circ$. Thus it remains to show that $(u^*Y)^\circ \leq u^*Y^\circ$. This follows since by adjointness it is equivalent to $u^*Y \leq \overline{u^*Y^\circ} = u^*\overline{Y^\circ} = u^*\overline{Y}$, which is true. □

Corollary 8.2.8. *Let $u\colon I \to J$ be a morphism of discrete objects in $\mathcal{E}$ and let $X \in \mathrm{Sub}_{\mathcal{E}}(I)$ be a subobject of I. Then $(\forall_u X^\circ)^\circ = (\forall_u X)^\circ$.*

Proof. Using Lemma 8.2.7 (twice) we get $(\forall_u X^\circ)^\circ = (\forall_u \overline{X^\circ})^\circ = (\forall_u \overline{X})^\circ = (\forall_u X)^\circ$. □

Using the two lemmas above the first-order logic of $\mathrm{D}_j\mathcal{E}$ (*i.e.*, the internal logic of the subobject fibration on $\mathrm{D}_j\mathcal{E}$) can be written out explicitly as follows. We label the connectives with a subscript d. For all objects (types) $I, J \in \mathrm{D}_j\mathcal{E}$, all subobjects in $\mathrm{D}_j\mathcal{E}$ (corresponding to open subobjects in $\mathcal{E}$) $X, Y \in \mathrm{Sub}_{\mathrm{D}_j\mathcal{E}}(I)$ and $Z \in \mathrm{Sub}_{\mathrm{D}_j\mathcal{E}}(I \times J)$, and morphisms (terms) $x, x'\colon 1 \to X$

$$\begin{aligned}
(x =_d x') &= (x = x') \\
\top_d &= \top \\
X \wedge_d Y &= (X \wedge Y) \\
\bot_d &= \bot \\
X \vee_d Y &= X \vee Y \\
X \supset_d Y &= (X \supset Y)^\circ \\
\neg_d X &= (X \supset \bot)^\circ \\
\exists_d j\colon J.\ Z &= \exists j\colon J.\ Z \\
\forall_d j\colon J.\ Z &= (\forall j\colon J.\ Z)^\circ
\end{aligned} \tag{8.3}$$

(Note that negation has also been simplified thanks to Lemma 8.2.6.)

Substitution in $\begin{array}{c}\mathrm{Sub}(\mathrm{D}_j\mathcal{E})\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ is interpreted by pullback in $\mathrm{D}_j\mathcal{E}$, which, when we view the types in $\mathcal{E}$, is the same as pullback in $\mathcal{E}$.

We see that the geometric part of the logic ($\top$, $\wedge$, $\bot$, $\vee$, $\exists$) is interpreted exactly as in $\mathcal{E}$. This should come as no surprise because the inclusion of the discrete objects into $\mathcal{E}$ is the inverse image of a geometric morphism and thus it preserves geometric logic.

8.2.1 Preservation of Validity of Stable Formulas

We now aim to show that a wider class of sentences than the class of geometric sentences is preserved by the inclusion of the discrete objects,

Let $\Gamma \vdash \varphi\colon \mathsf{Prop}$ be a formula of first-order logic over a first-order many-sorted language. Suppose that the basic types of the language are interpreted in $\mathcal{E}$ by discrete objects and that the atomic predicates are interpreted by open subobjects of discrete objects in $\mathcal{E}$, corresponding to subobjects in $\mathrm{D}_j\mathcal{E}$. We then write $[\![\Gamma \vdash \varphi\colon \mathsf{Prop}]\!]$ for the interpretation of $\Gamma \vdash \varphi\colon \mathsf{Prop}$ in $\mathcal{E}$, *i.e.*, in the subobject fibration over $\mathcal{E}$. Likewise, we write $[\![\Gamma \vdash \varphi\colon \mathsf{Prop}]\!]_d$ for the interpretation of $\Gamma \vdash \varphi\colon \mathsf{Prop}$ in $\mathrm{D}_j\mathcal{E}$, *i.e.*, in the subobject fibration over $\mathrm{D}_j\mathcal{E}$. For notational simplicity we often abbreviate and write $[\![\varphi]\!]$ for $[\![\Gamma \vdash \varphi\colon \mathsf{Prop}]\!]$ and $[\![\varphi]\!]_d$ for $[\![\Gamma \vdash \varphi\colon \mathsf{Prop}]\!]_d$. Moreover, we allow ourselves to consider $[\![\varphi]\!]_d \in \mathrm{Sub}_{\mathrm{D}_j\mathcal{E}}([\![\Gamma]\!]_d)$ as a subobject in $\mathcal{E}$, thus eliding the inclusion functor from discrete objects into $\mathcal{E}$ (here $[\![\Gamma]\!]_d$ denotes the discrete object interpreting Γ). Finally, we say that $\Gamma \vdash \varphi\colon \mathsf{Prop}$ is valid in $\mathcal{E}$, written in short as $\mathcal{E} \vDash \varphi$, iff $\top \leq [\![\varphi]\!]$ in $\mathrm{Sub}_{\mathcal{E}}([\![\Gamma]\!])$, where $[\![\Gamma]\!]$ is the interpretation of Γ.[1] Likewise, we say that $\Gamma \vdash \varphi\colon \mathsf{Prop}$ is valid in $\mathrm{D}_j\mathcal{E}$, written $\mathrm{D}_j\mathcal{E} \vDash \varphi$, if $\top_d \leq [\![\varphi]\!]_d$ in $\mathrm{Sub}_{\mathrm{D}_j\mathcal{E}}([\![\Gamma]\!]_d)$.

Definition 8.2.9. Let $\Gamma \vdash \varphi\colon \mathsf{Prop}$ be a formula of first-order logic over a first-order many-sorted language. We say that φ is **stable** if, for all subformulas $(\psi \supset \vartheta)$ of φ, the formula ψ is geometric.

Lemma 8.2.10. *Let $\Gamma \vdash \varphi\colon \mathsf{Prop}$ be a formula of first-order logic over a first-order many-sorted language. If φ is stable, then $[\![\varphi]\!]^\circ = [\![\varphi]\!]_d$.*

Proof. The proof is by structural induction on φ. Note that $[\![\varphi]\!]_d$ is discrete, and thus open, so ${[\![\varphi]\!]_d}^\circ = [\![\varphi]\!]_d$. For φ atomic we clearly have $[\![\varphi]\!] = [\![\varphi]\!]_d$ and thus also $[\![\varphi]\!]^\circ = [\![\varphi]\!]_d$. Given the result for atomic formulas, for φ a geometric formula, we clearly also find that $[\![\varphi]\!] = [\![\varphi]\!]_d$ — see the explicit description of the logical operations in $\begin{array}{c}\mathrm{Sub}(\mathrm{D}_j\mathcal{E})\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ on Page 183. Hence also $[\![\varphi]\!]^\circ = [\![\varphi]\!]_d$. It remains to consider implication and universal quantification.

[1] Take note that $\mathcal{E} \vDash \varphi$ refers to the interpretation $[\![\varphi]\!]$ where atomic predicates are interpreted by open subobjects.

Suppose that $\varphi \equiv \psi \supset \vartheta$. Then we have that

$$\begin{aligned}
[\![\psi \supset \vartheta]\!]_d &= ([\![\psi]\!]_d \supset [\![\vartheta]\!]_d)^\circ && \text{see definition of } \supset_d \text{ on Page 183} \\
&= ([\![\psi]\!]^\circ \supset [\![\vartheta]\!]^\circ)^\circ && \text{by induction hypothesis} \\
&= ([\![\psi]\!]^\circ \supset [\![\vartheta]\!])^\circ && \text{by Lemma 8.2.6} \\
&= ([\![\psi]\!] \supset [\![\vartheta]\!])^\circ && \text{since } \psi \text{ is geometric by stability of } \varphi,
\end{aligned}$$

as required.

Finally, suppose that $\varphi \equiv \forall x \colon X.\ \psi$. Then we have that

$$\begin{aligned}
[\![\forall x \colon X.\ \psi]\!]_d &= (\forall x \colon X.\ [\![\psi]\!]_d)^\circ && \text{see definition of } \forall_d \text{ on Page 183} \\
&= (\forall x \colon X.\ [\![\psi]\!])^{\circ\circ} && \text{by induction} \\
&= (\forall x \colon X.\ [\![\psi]\!])^\circ && \text{by Corollary 8.2.8} \\
&= [\![\forall x \colon X.\ \psi]\!]^\circ,
\end{aligned}$$

as required. □

Theorem 8.2.11. *Let* $\Gamma \vdash \varphi\colon \mathsf{Prop}$ *be a formula of first-order logic over a first-order many-sorted language. If* $\Gamma \vdash \varphi\colon \mathsf{Prop}$ *is stable, then* $\mathcal{E} \models \varphi$ *iff* $\mathrm{D}_j\mathcal{E} \models \varphi$.

Proof. Let $I = [\![\Gamma]\!] = [\![\Gamma]\!]_d$ be the discrete object interpreting Γ. Then, writing $\leq_d$ for the ordering in $\mathrm{Sub}_{\mathrm{D}_j\mathcal{E}}(I)$ and writing $\leq$ for the ordering in $\mathrm{Sub}_{\mathcal{E}}(I)$, we have that

$$\begin{aligned}
& \mathrm{D}_j\mathcal{E} \models \varphi \\
\iff\ & \top_d \leq_d [\![\varphi]\!]_d \\
\iff\ & \top \leq [\![\varphi]\!]_d && \text{since } \top_d = \top, \text{ see Page 183} \\
\iff\ & \top \leq [\![\varphi]\!]^\circ && \text{by Lemma 8.2.10} \\
\iff\ & \top \leq [\![\varphi]\!] && \text{since } I \text{ is discrete and thus open} \\
\iff\ & \mathcal{E} \models \varphi.
\end{aligned}$$

□

8.3 A Modal Logic for Local Maps

So far in this chapter we have only used the interior operation as a semantical operation; we have not considered it syntactically as a logical operator. We do so in this section. As explained in the introduction to this chapter, interior is not a logical operation in the subobject fibration over $\mathcal{E}$ because it does not commute with substitution. However, when we restrict attention to discrete objects, interior *does* commute with substitution, see Proposition 8.3.2 below.

The following definition makes precise the idea of considering the logic of $\mathcal{E}$ restricted to discrete objects.

Definition 8.3.1. We define the fibration $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ of $\mathcal{E}$-predicates over $\mathrm{D}_j\mathcal{E}$ by change-of-base as in

$$\begin{tikzcd}
\mathrm{Pred} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sub}(\mathcal{E}) \arrow[d] \\
\mathrm{D}_j\mathcal{E} \arrow[r, hook] & \mathcal{E}.
\end{tikzcd}$$

Thus in the internal logic of $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$, types and terms are interpreted by objects and morphisms of $\mathrm{D}_j\mathcal{E}$ and predicates over a type σ, interpreted by a discrete object I, are interpreted as subobjects of I in $\mathcal{E}$. In other words, we consider all the predicates of $\mathcal{E}$, but only on types and terms from $\mathrm{D}_j\mathcal{E}$.

The fibration $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ is clearly a first-order fibration. (In general it does not have a generic object since the subobject classifier Ω in $\mathcal{E}$ in general is not a discrete object). We now show that interior commutes with substitution (reindexing) in $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$.

Proposition 8.3.2. *Let $u\colon I \to J$ be a morphism between discrete objects I and J in $\mathcal{E}$ and suppose $X \rightarrowtail J$ is a subobject of J. Then $(u^*X)^\circ = u^*(X^\circ)$ as subobjects of I.*

Proof. First note that by Proposition 7.4.7, $u^*(X^\circ)$ is open. Thus $u^*(X^\circ) = u^*(X^\circ)^\circ \leq u^*X^\circ$. The other direction always holds (regardless of I and J being discrete): $(u^*X)^\circ \leq u^*(X^\circ)$ iff $u^*X \leq \overline{u^*(X^\circ)} = u^*\overline{X}$. $\square$

It is instructive to note that the above proposition can also be seen as a corollary to the following observation:

Proposition 8.3.3. *Let I be a discrete object and let $X \rightarrowtail I$ be a subobject of I in $\mathcal{E}$. Then the interior X° of X is $\Delta\Gamma X$ (up to isomorphism).*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} \Delta\Gamma X \arrow[r, "\epsilon_X"] \arrow[d, tail, "\Delta\Gamma m"'] & X \arrow[d, tail, "m"] \\ \Delta\Gamma I \arrow[r, "\epsilon_I"'] & I, \end{tikzcd}$$

where ϵ_I and ϵ_X are the counits. Since I is discrete the arrow ϵ_I across the bottom is iso. Since Δ and Γ are both left exact, $\Delta\Gamma m$ is monic. Thus $m \circ \epsilon_X = \epsilon_I \circ \Delta\Gamma m$ is monic, and hence ϵ_X is monic. Therefore $X^\circ = \mathrm{Im}(\epsilon_X) \cong \Delta\Gamma X$. $\square$

Now Proposition 8.3.2 is obtained from Proposition 8.3.3 simply by recalling that Δ and Γ are left exact and thus commute with pullback.

By Proposition 8.3.2, the interior operation is a logical operation in $\begin{array}{c}\mathrm{Pred}\\ \downarrow \\ \mathrm{D}_j\mathcal{E}\end{array}$. So is, of course, also the closure operation. In the next subsection we describe how the interior and closure operations can be axiomatized, so as to obtain what we will refer to as a *modal logic for local maps.* In the syntactic calculus we denote interior by $\sharp$ and closure by $\flat$. The choice of this notation comes from our use of $\sharp$ in our realizability model $\mathrm{RT}(A, A_\sharp)$. In Subsection 8.3.2 we prove that $\begin{array}{c}\mathrm{Pred}\\ \downarrow \\ \mathrm{D}_j\mathcal{E}\end{array}$, as expected, provides a model for the syntactic calculus.

8.3.1 Axiomatization of a Modal Logic for Local Maps

We describe an extension of standard intuitionistic first-order logic. As usual we write logical entailment as $\Gamma \mid \Theta \vdash \psi$, where Γ is a context of the form $x_1 : \sigma_1, \ldots, x_n : \sigma_n$ giving types σ_i to variables x_i, and where ψ is a formulas with free variables in Γ, and Θ is a list of formulas with free variables in Γ. We write $\emptyset$ for an empty list of assumptions. There are two additional logical operations: if φ is a formula, also $\sharp\varphi$ and $\flat\varphi$ are formulas. Substitution of terms for variables in these new formulas is defined in the obvious way:

$$(\sharp\varphi)[M/x] \equiv \sharp(\varphi[M/x]) \qquad \text{and} \qquad (\flat\varphi)[M/x] \equiv \flat(\varphi[M/x]).$$

There are the usual rules of many-sorted first-order intuitionistic logic plus the following axioms and rules:

$$\frac{}{\Gamma \mid \sharp\varphi \vdash \varphi}\,(8.4) \qquad \frac{}{\Gamma \mid \sharp\varphi \vdash \sharp\sharp\varphi}\,(8.5)$$

$$\frac{}{\Gamma \mid \emptyset \vdash \sharp(\top)}\,(8.6) \qquad \frac{}{\Gamma \mid \sharp\varphi \wedge \sharp\psi \vdash \sharp(\varphi \wedge \psi)}\,(8.7)$$

$$\frac{\Gamma \mid \sharp\varphi \vdash \psi}{\Gamma \mid \varphi \vdash \flat\psi}\,(8.8) \qquad \frac{}{x\colon \sigma, y\colon \sigma \mid x =_\sigma y \vdash \sharp(x =_\sigma y)}\,(8.9)$$

Note that Rule (8.7) is a double-rule which can be applied in both directions. Intuitively, Axiom (8.4) says that $\sharp$ is a deflationary operation, Axiom (8.5) then says that $\sharp$ is idempotent, Axioms (8.6) and (8.7) say that $\sharp$ is left exact, Rule (8.7) says that $\sharp$ is left adjoint to $\flat$, and Axiom (8.9) expresses that all the types are discrete and hence equality is $\sharp$.

From the above axioms and rules one can derive that all of the following hold (for notational simplicity, we here leave out the context Γ, which does not change):

$$\frac{\varphi \vdash \psi}{\sharp\varphi \vdash \sharp\psi}\,(8.10) \qquad \frac{\varphi \vdash \psi}{\flat\varphi \vdash \flat\psi}\,(8.11)$$

$$\frac{}{\sharp(\varphi \wedge \psi) \vdash \sharp\varphi \wedge \sharp\psi}\,(8.12) \qquad \frac{}{\sharp(\varphi \supset \psi) \vdash \sharp\varphi \supset \sharp\psi}\,(8.13)$$

$$\frac{}{\varphi \vdash \flat\varphi}\,(8.14) \qquad \frac{}{\flat\varphi \dashv\vdash \flat\flat\varphi}\,(8.15)$$

$$\frac{}{\flat\top \vdash \top}\,(8.16) \qquad \frac{}{\flat(\varphi \wedge \psi) \dashv\vdash \flat\varphi \wedge \flat\psi}\,(8.17)$$

$$\frac{\emptyset \vdash \varphi}{\emptyset \vdash \sharp\varphi}\,(8.18)$$

$$\frac{}{\sharp\flat\varphi \dashv\vdash \sharp\varphi}\,(8.19) \qquad \frac{}{\flat\sharp\varphi \dashv\vdash \flat\varphi}\,(8.20)$$

The proofs of the above rules are obtained simply by formalizing the usual categorical proofs. For example, (8.10) is proved as follows:

$$\dfrac{\dfrac{\varphi \vdash \psi \qquad \dfrac{\dfrac{}{\sharp\psi \vdash \sharp\psi}\,\textbf{identity}}{\psi \vdash \flat\sharp\psi}\,(8.8)}{\varphi \vdash \flat\sharp\psi}\,\textbf{cut}}{\sharp\varphi \vdash \sharp\psi}\,(8.8)$$

As another example, (8.18) is proved as follows:

$$\dfrac{\dfrac{}{\top \vdash \sharp\top}\,(8.6) \qquad \dfrac{\top \vdash \varphi}{\sharp\top \vdash \sharp\varphi}\,(8.10)}{\top \vdash \sharp\varphi}\,\textbf{cut}$$

Note that (8.4), (8.5), (8.6), and (8.18) together express that $\sharp$ has the formal properties of the box operator in the modal logic S4. That is why we refer to the first-order logic axiomatized in this section as a *modal logic for local maps*. It should be observed that $\flat$ is *not* the usual diamond, however, which is (classically) *left* rather than right adjoint to box.

The following principles of inference for the quantifiers can be derived:

$$\dfrac{\Gamma \mid \emptyset \vdash \sharp(\forall x\colon \sigma.\ \varphi)}{\Gamma \mid \emptyset \vdash \forall x\colon \sigma.\ \sharp\varphi}\,(8.21) \qquad \dfrac{}{\Gamma \mid \sharp(\exists x\colon \sigma.\ \varphi) \dashv\vdash \exists x\colon \sigma.\ \sharp\varphi}\,(8.22)$$

(the rule (8.21) is printed with a double line)

Rule (8.21) is derived as follows (using Rules (8.4) and (8.18)):

$$\begin{array}{c} \Gamma \mid \top \vdash \forall x\colon \sigma.\ \varphi \\ \hline\hline \Gamma, x\colon \sigma \mid \top \vdash \varphi \\ \hline\hline \Gamma, x\colon \sigma \mid \top \vdash \sharp\varphi \\ \hline\hline \Gamma \mid \top \vdash \forall x\colon \sigma.\ \sharp\varphi \end{array}$$

(Here we are also using the so-called mate-rule for $\forall$:

$$\begin{array}{c} \Gamma \mid \Theta, \varphi \vdash \forall x\colon \sigma.\ \psi \\ \hline\hline \Gamma, x\colon \sigma \mid \Theta, \varphi \vdash \psi \end{array}$$

See, *e.g.*, [Jac99, Lemma 4.1.8, Page 230] for the derivation of this double-rule.)

The equivalence in Rule (8.22) is proved as follows. First note that, for any $\Gamma \vdash \psi$ we have:

$$\begin{array}{c} \Gamma \mid \sharp(\exists x\colon \sigma.\ \varphi) \vdash \psi \\ \hline\hline \Gamma \mid \exists x\colon \sigma.\ \varphi \vdash \flat\psi \\ \hline\hline \Gamma, x\colon \sigma \mid \varphi \vdash \flat\psi \\ \hline\hline \Gamma, x\colon \sigma \mid \sharp\varphi \vdash \psi \\ \hline\hline \Gamma \mid \exists x\colon \sigma.\ \sharp\varphi \vdash \psi \end{array}$$

(Here we are also using the so-called mate-rule for $\exists$:

$$\frac{\Gamma \mid \Theta, \exists x\colon \sigma.\ \varphi \vdash \psi}{\overline{\Gamma, x\colon \sigma \mid \Theta, \varphi \vdash \psi}}$$

See, *e.g.*, [Jac99, Lemma 4.1.8, Page 230] for the derivation of this double-rule.) Hence by plugging in $\exists x\colon \sigma.\ \sharp\varphi$ for ψ, we get $\sharp(\exists x\colon \sigma.\ \varphi) \vdash \exists x\colon \sigma.\ \sharp\varphi$. Likewise, by plugging in $\sharp(\exists x\colon \sigma.\ \varphi)$ for ψ we get $\exists x\colon \sigma.\ \sharp\varphi \vdash \sharp(\exists x\colon \sigma.\ \varphi)$, thus completing the proof of Rule (8.22).

8.3.2 A Model for the Modal Logic for Local Maps

Proposition 8.3.4. *The fibration* $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ *is a model for the modal logic for local maps.*

Proof. The interpretation of types and terms is given in the standard way. with $\sharp$ interpreted by the interior operation and $\flat$ interpreted by the closure operation. Substitution is interpreted correctly by Proposition 8.3.2. The standard first-order logic is interpreted soundly since $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ is a first-order fibration. The new axioms and rules of the modal logic for local maps clearly validated: Rules (8.4) and (8.5) hold by the fact that interior is deflationary and idempotent, see Definition 7.3.3 and Lemma 7.3.5. Rule (8.6) is sound since $\top$ is interpreted by the maximal subobject of the discrete object I interpreting Γ — I being discrete entails that I is open and thus that $I \leq_I I^\circ$ in $\mathrm{Pred}_I = \mathrm{Sub}_{\mathcal{E}}(I)$. Rule (8.7) is sound because supposing that Γ is interpreted by discrete object I and that φ and ψ are interpreted by X and Y in Pred_I, respectively, then X° and Y° are discrete (as open subobjects of a discrete object), so the pullback $X^\circ \wedge Y^\circ$ is discrete, and thus open, so $X^\circ \wedge Y^\circ = (X^\circ \wedge Y^\circ)^\circ \leq (X \wedge Y)^\circ$. For Rule (8.9) note that the equalizer of a pair of morphisms between discrete objects in $\mathcal{E}$ is discrete, since the discrete objects are closed under finite limits. Hence the equality predicate of two terms is interpreted by a discrete and hence open subobject. □

8.3.3 Applications of the Modal Logic for Local Maps

In this subsection we present a couple of examples of applications of the logic of discrete objects and the modal logic for local maps.

Observe that the definition of the logical predicates in $\begin{array}{c}\mathrm{Sub}(\mathrm{D}_j\mathcal{E})\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ in (8.3) on Page 183 can now be seen as a *syntactic translation* from formulas φ

of first-order logic into formulas of the modal logic for local maps. We write $|\varphi|$ for the translation. It is defined in the obvious way — view (8.3) syntactically and replace the interior operation in the defining equations with a $\sharp$. Explicitly:

$$\begin{aligned} |(x = x')| &= (x = x') \\ |\top| &= \top \\ |\varphi \wedge \psi| &= |\varphi| \wedge |\psi| \\ |\bot| &= \bot \\ |\varphi \vee \psi| &= |\varphi| \vee |\psi| \\ |\varphi \supset \psi| &= \sharp(|\varphi| \supset |\psi|) \\ |\neg\varphi| &= \sharp(|\varphi| \supset \bot) \\ |\exists j\colon J.\ \varphi| &= \exists j\colon J.\ |\varphi| \\ |\forall j\colon J.\ \varphi| &= \sharp(\forall j\colon J.\ |\varphi|) \end{aligned} \tag{8.23}$$

Note that for φ a geometric formula, $|\varphi| = \varphi$.

In the following discussion it will be convenient to assume that our basic language of formulas contains two kinds of relation symbols: R, S, ... , and $R^\sharp$, $S^\sharp$, ... When considering a first-order formula φ and its interpretation in $\begin{array}{c}\mathrm{Sub}(\mathcal{E})\\ \downarrow\\ \mathcal{E}\end{array}$ we will assume that the relational symbol $R^\sharp$ is interpreted by the interior of the interpretation of the relational symbol R. Likewise when considering φ a formula of the modal logic for local maps. Finally, when considering φ a formula of the logic of $\begin{array}{c}\mathrm{Sub}(\mathrm{D}_j\mathcal{E})\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$, the formula φ must not contain any relational symbols of the form R (*i.e.*, all relational symbols must be of the form $R^\sharp$). The translation given above is defined on atomic relational symbols as follows:

$$|R^\sharp| = \sharp R \qquad \text{and} \qquad |R| = R.$$

We write[2]

- $[\![\varphi]\!]$ for the interpretation of φ in $\begin{array}{c}\mathrm{Sub}(\mathcal{E})\\ \downarrow\\ \mathcal{E}\end{array}$
- $[\![\varphi]\!]_m$ for the interpretation of φ in $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$

[2]As on Page 184 we really interpret sequents $\Gamma \vdash \varphi$: Prop, but we shall not make that explicit in this section.

- $[\![\varphi]\!]_d$ for the interpretation of φ in $\begin{array}{c}\mathrm{Sub}(\mathrm{D}_j\mathcal{E})\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$

and we write

- $\mathcal{E} \models \varphi$ if $[\![\varphi]\!]$ is valid in $\begin{array}{c}\mathrm{Sub}(\mathcal{E})\\ \downarrow\\ \mathcal{E}\end{array}$
- $\mathrm{Pred} \models \varphi$ if $[\![\varphi]\!]_m$ is valid in $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$
- $\mathrm{D}_j\mathcal{E} \models \varphi$ if $[\![\varphi]\!]_d$ is valid in $\begin{array}{c}\mathrm{Sub}(\mathrm{D}_j\mathcal{E})\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$

We then have that

1. If φ a formula of first-order logic in which all relational symbols are of the form $R^\sharp$ and if all the basic types in φ are interpreted by discrete objects, then $\mathrm{D}_j\mathcal{E} \models \varphi$ iff $\mathrm{Pred} \models |\varphi|$.
2. If φ is a formula of first-order logic and all the basic types in φ are interpreted by discrete objects, then $\mathcal{E} \models \varphi$ iff $\mathrm{Pred} \models \varphi$.

Using the conventions established here, we now consider two sample applications of the modal logic for local maps.

External Axiom of Choice

For objects I and J in an arbitrary topos $\mathcal{E}$, let us say that *the (external) axiom of choice holds from I to J*, written $\mathrm{EAC}(X, Y)$, if, for any subobject $R \rightarrowtail I \times J$, if $\forall i\colon I.\ \exists j\colon J.\ R(i,j)$ is valid (in the subobject fibration over $\mathcal{E}$) then there exists a morphism $f\colon I \to J$ in such that $\forall i\colon I.\ R(i, f(i))$ is valid (in the subobject fibration over $\mathcal{E}$).

Proposition 8.3.5. *Let I and J be discrete objects and suppose that the external axiom of choice holds from I to J in $\mathrm{D}_j\mathcal{E}$. Then also the external axiom of choice holds from I to J in $\mathcal{E}$.*

Proof. We argue as follows:

$$\begin{array}{lll}
 & \mathcal{E} \models \forall i\colon I.\ \exists j\colon J.\ R(i,j) & \\
\Longrightarrow & \mathrm{Pred} \models \forall i\colon I.\ \exists j\colon J.\ R(i,j) & \\
\Longrightarrow & \mathrm{Pred} \models \sharp(\forall i\colon I.\ \exists j\colon J.\ \sharp R(i,j)) & \text{by modal logic} \\
\Longrightarrow & \mathrm{Pred} \models |\forall i\colon I.\ \exists j\colon J.\ R^\sharp(i,j)| & \\
\Longrightarrow & \mathrm{D}_j\mathcal{E} \models \forall i\colon I.\ \exists j\colon J.\ R^\sharp(i,j) &
\end{array}$$

so by EAC(I, J) in $\mathrm{D}_j\mathcal{E}$, there exists an $f: I \to J$ such that

$$\begin{array}{lll}
\Longrightarrow & \mathrm{D}_j\mathcal{E} \models \forall i\colon I.\ R^\sharp(i, f(i)) & \\
\Longrightarrow & \mathrm{Pred} \models \sharp(\forall i\colon I.\ \sharp R(i, f(i))) & \\
\Longrightarrow & \mathrm{Pred} \models \forall i\colon I.\ R(i, f(i)) & \text{by modal logic} \\
\Longrightarrow & \mathcal{E} \models \forall i\colon I.\ R(i, f(i)). &
\end{array}$$

□

Church's Thesis

In this example we show that if the topos of discrete objects satisfies the arithmetic form of Church's Thesis (in the sense of, *e.g.*, [TvD88, Tro73]), then a $\sharp$'ed version is satisfied by $\mathcal{E}$.

Observe that $\mathcal{E}$ has a natural numbers object if and only if $\mathrm{D}_j\mathcal{E}$ has a natural numbers object, because both $\Delta\colon \mathrm{D}_j\mathcal{E} \to \mathcal{E}$ and $\Gamma\colon \mathcal{E} \to \mathrm{D}_j\mathcal{E}$ are inverse images of geometric morphisms and as such they preserve the natural numbers object (see, *e.g.*, [Joh77, Proposition 6.12]).

For the remainder of this subsection we assume that $\mathrm{D}_j\mathcal{E}$ has a natural numbers object N which thus also is the natural numbers object of $\mathcal{E}$.

Recall from Kleene's Normal Form Theorem that the basic predicates of recursion theory can be defined from Kleene's T-predicate and output function $U\colon N \to N$, see, *e.g.*, [TvD88]. The predicates $T \rightarrowtail N \times N \times N$ and $U(-) = (-)$ are both primitive recursive. Hence their interpretation is preserved by the inclusion $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ (*i.e.*, the interpretation of T in $\mathrm{D}_j\mathcal{E}$ agrees with the interpretation of T in $\mathcal{E}$). Indeed, a predicate R on N is primitive recursive iff there is a primitive recursive function $\chi_R\colon N \to N$ such that $\chi_R(n) = 1$ iff $R(n)$ holds. Hence to show that the interpretation of primitive recursive predicates is preserved by the inclusion $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$, it suffices to note that equality is preserved, which it is (see $=_d$ on Page 183), and that the interpretation of primitive recursive functions is preserved. The latter holds as expressed by the following lemma.

Lemma 8.3.6. *Let $n\colon \mathsf{N} \vdash M\colon \mathsf{N}$ be a term built up from the clauses defining the class of primitive recursive functions (see,* e.g., *[TvD88, Definition 3.1.2]). Then the interpretation of $n\colon \mathsf{N} \vdash M\colon \mathsf{N}$ in the topos $\mathrm{D}_j\mathcal{E}$ (a morphism from N to N in $\mathrm{D}_j\mathcal{E}$) is the same as the interpretation in $\mathcal{E}$.*

Proof. By induction on the construction of the term M. For the zero constant, the successor, and the projections, the result follows since the natural numbers object N in $\mathrm{D}_j\mathcal{E}$ is also the natural numbers object in $\mathcal{E}$ and because the discrete objects are closed under products in $\mathcal{E}$. For composition, the result follows since $\mathrm{D}_j\mathcal{E}$ is a full *subcategory* (so closed under composition in $\mathcal{E}$). For definition by recursion, we use the formulation of a natural numbers object involving parameters (see, *e.g.*, [LS86, Exercise 9.4]): The clause for definition by recursion says that if $f\colon N^n \to N$ and $g\colon N^{n+2} \to N$ are primitive recursive, then there is a primitive recursive function $h\colon N^n \to N$ such that

$$\begin{aligned} h(0, x_1, \ldots, x_n) &= f(x_1, \ldots x_n) \\ h(Sy, x_1, \ldots, x_n) &= g(h(y, x_1, \ldots x_n), y, x_1, \ldots x_n). \end{aligned}$$

Given interpretations $f\colon N^n \to N$ and $g\colon N^{n+2} \to N$ of primitive recursive terms $f\colon \mathsf{N}^n \to \mathsf{N}$ and $g\colon \mathsf{N}^{n+2} \to \mathsf{N}$, the interpretation of the primitive recursive term $h\colon \mathsf{N}^n \to \mathsf{N}$ is $h = \pi \circ k$, where k is the unique morphism making the diagram below commute

$$\begin{tikzcd} N^n \arrow[r, "\langle 0, id\rangle"] \arrow[d, equal] & N \times N^n \arrow[r, "\langle S, id\rangle"] \arrow[d, "k"] & N \times N^n \arrow[d, "k"] \\ N^n \arrow[r, "\langle f, 0, id\rangle"'] & N \times N \times N^n \arrow[r, "\langle g, S, id\rangle"'] & N \times N \times N^n. \end{tikzcd}$$

Since this is a diagram only involving discrete objects, the interpretation in $\mathrm{D}_j\mathcal{E}$ and in $\mathcal{E}$ is of course the same. $\square$

In summary, T and U are interpreted in the same way in $\mathcal{E}$ as in $\mathrm{D}_j\mathcal{E}$.

We recall the following definition from [TvD88, Section 4.3].

Definition 8.3.7. The **arithmetical form of Church's Thesis** is the schema

$$\mathrm{CT}_0 \qquad \forall n.\ \exists m.\ \varphi(n,m) \supset \exists k.\ \forall n.\ \exists m.\ \big(\varphi(n, Um) \wedge T(k,n,m)\big),$$

where all the variables range over the type of natural numbers N and where φ is a formula.

Proposition 8.3.8. *Let φ be a formula and suppose that the φ instance of CT_0 holds in* $\begin{smallmatrix}\mathrm{Sub}(\mathrm{D}_j\mathcal{E})\\ \downarrow \\ \mathrm{D}_j\mathcal{E}\end{smallmatrix}$. *Then the following formula holds in* $\begin{smallmatrix}\mathrm{Pred}\\ \downarrow \\ \mathrm{D}_j\mathcal{E}\end{smallmatrix}$*:*

$$\sharp(\forall n.\ \exists m.\ |\varphi|(n,m)) \supset \exists k.\ \forall n.\ \exists m.\ \big(|\varphi|(n, Um) \wedge T(k,n,m)\big).$$

Proof. By the fact that $\mathrm{D}_j\mathcal{E} \vDash \varphi$ iff $\mathrm{Pred} \vDash |\varphi|$ we get that

$$\sharp(\forall n.\ \exists m.\ |\varphi|(n,m)) \supset \exists k.\ \sharp\forall n.\ \exists m.\ \big(|\varphi|(n,Um) \wedge T(k,n,m)\big)$$

holds in $\underset{\mathrm{D}_j\mathcal{E}}{\overset{\mathrm{Pred}}{\downarrow}}$. The required then follows by the modal logic, using that $\sharp$ commutes with $\exists$ and that $\sharp(\sharp\varphi \supset \psi) \dashv\vdash \sharp\varphi \supset \sharp\psi$. $\square$

Thus, in particular, if all geometric instances of CT_0 hold in $\mathrm{D}_j\mathcal{E}$, then all geometric instances of

$$\mathrm{CT}_0^\sharp \qquad \sharp(\forall n.\ \exists m.\ \varphi(n,m)) \supset \exists k.\ \forall n.\ \exists m.\ \big(\varphi(n,Um) \wedge T(k,n,m)\big)$$

hold in $\underset{\mathrm{D}_j\mathcal{E}}{\overset{\mathrm{Pred}}{\downarrow}}$.

Chapter 9

Logic and Localic Local Maps of Toposes

Suppose given a topos $\mathcal{E}$ with a topology j satisfying the axioms for *localic* local maps set out in the Chapter 7. There results a localic local map of toposes

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\Gamma"] \\ \mathrm{D}_j\mathcal{E} \arrow[u, bend left, "\Delta"] \arrow[u, bend right, "\nabla"'] \end{tikzcd} \qquad \Delta \dashv \Gamma \dashv \nabla$$

with (Δ, Γ) the localic local map, Δ the inclusion of the discrete objects and Γ the associated discrete object functor with right adjoint ∇. All the results from the previous chapter applies, since a localic local map is of course a special case of a bounded local map.

In this chapter we investigate two additional simple points of view that result from the extra assumption that the local map is localic. In Section 9.1 we take the point of view of tripos theory and show that the modal logic resulting from the localic local map is just a particular case of tripos logic. We define a notion of *local tripos* and show that any local tripos gives rise to a localic local map of toposes and, moreover, that any localic local map of toposes comes from a local tripos. The actual tripos that results from a localic local map is naturally one given on an internal locale (complete Heyting algebra). In Section 9.2 we take the point of view of internal locale theory and describe the modal operators as certain easily given internal maps on an internal locale. We further observe that a substantial part of the modal logic follows from very weak assumptions (whenever one has an internal locale in some topos).

9.1 Local Triposes

Consider the fibration $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ obtained in the previous chapter by change-of-base as in

$$\begin{tikzcd}
\mathrm{Pred} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sub}(\mathcal{E}) \arrow[d] \\
\mathrm{D}_j\mathcal{E} \arrow[r, hook] & \mathcal{E}.
\end{tikzcd}$$

Note that, for any $I \in \mathrm{D}_j\mathcal{E}$, we have that

$$\mathrm{Pred}_I = \mathrm{Sub}_{\mathcal{E}}(\Delta I) \cong \mathcal{E}(\Delta I, \Omega_{\mathcal{E}}) \cong \mathrm{D}_j\mathcal{E}(I, \Gamma\Omega_{\mathcal{E}}) \tag{9.1}$$

where $\Omega_{\mathcal{E}}$ is the subobject classifier in $\mathcal{E}$ and where the isomorphisms are natural in I.

Since $\mathcal{E}$ is localic over $\mathrm{D}_j\mathcal{E}$ via (Δ, Γ), $\mathcal{E}$ is the topos of $\mathrm{D}_j\mathcal{E}$-valued sheaves on the internal locale $\Gamma\Omega_{\mathcal{E}}$ in $\mathrm{D}_j\mathcal{E}$ [Joh77, Joh81] In other words, $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ is (equivalent to) the canonical $\mathrm{D}_j\mathcal{E}$-tripos on the internal locale $\Gamma\Omega_{\mathcal{E}}$ and the modal internal logic of $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ is a particular example of tripos logic. Moreover, $\mathcal{E}$ is the topos obtained by the tripos-to-topos construction applied to the tripos $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$.

We now define a notion of local tripos and show that any local tripos gives rise to a localic local map of toposes and, moreover, that any localic local map of toposes comes from a local tripos. In the following section we take the viewpoint of the internal locale theory, and consider a notion of local internal locale. The internal locale theory point of view is perhaps more standard. Thus one may reasonably ask: what is the advantage of considering a notion of local tripos to describe a localic local map? The answer is that it can be much easier to *recognize* a local tripos rather than a local internal locale. We shall return to this point and make it clearer in the following.

Definition 9.1.1. Let $\begin{array}{c}\mathbb{P}\\ \downarrow\mathsf{p}\\ \mathcal{F}\end{array}$ be a canonically-presented tripos on an object Σ in a topos $\mathcal{F}$. The tripos p is said to be **local** if it comes together with maps $I, J\colon \Sigma \to \Sigma$ in $\mathcal{F}$ satisfying that

1. $p\colon \Sigma \mid Ip \vdash p$
2. $p\colon \Sigma \mid Ip \vdash IIp$
3. $\emptyset \mid \emptyset \vdash I(\top)$
4. $p, q\colon \Sigma \mid Ip \wedge Iq \vdash I(p \wedge q)$
5. $\dfrac{p, q\colon \Sigma \mid Ip \vdash q}{p, q\colon \Sigma \mid p \vdash Jq}$

all hold in the logic of p.

Note that the axioms and rules that have to hold for I and J are just as for $\sharp$ and $\flat$ in the modal logic for local maps in Section 8.3.1.[1] In other words, a local tripos models the modal logic for local maps, and we shall feel free to use the modal logic in the following when reasoning about local triposes.

Proposition 9.1.2. *Let $\mathsf{p} = \mathcal{F}(-, \Sigma)$ be a local tripos* qua $I, J\colon \Sigma \to \Sigma$. *Then J is a Lawvere-Tierney topology on p.*

Proof. One first shows from item (5) in the definition of local tripos that I and J are functorial. Then since I is deflationary, J is seen to be inflationary as a right adjoint to I. Moreover, since I is idempotent, J is also idempotent, again using adjointness. Finally, J preserves limits as a right adjoint. Further details are left to the reader. □

Let $\mathsf{p} = \mathcal{F}(-, \Sigma)$ be a local $\mathcal{F}$-tripos qua $I, J\colon \Sigma \to \Sigma$. We define a new canonically presented $\mathcal{F}$-tripos p_I as follows (in the following we prove that p_I so defined indeed is a tripos). Let $I\Sigma$ be the image of I in $\mathcal{F}$. Tripos p_I is canonically presented on $I\Sigma$. The ordering is defined as in p, that is, for $\varphi, \psi \in \mathcal{F}(X, I\Sigma)$ (p_I's fibre over $X \in \mathcal{F}$), we have $\varphi \vdash^{\mathsf{p}_I} \psi$ iff $\varphi \vdash^{\mathsf{p}} \psi$.

Since J is a topology by Proposition 9.1.2 we have a well-defined $\mathcal{F}$-tripos p_J as in Section 5.5.

[1]The only exception is that we in the definition of local tripos have left out the rule for equality – the rule for equality follows since equality in a tripos is given using existential quantification and truth $\top$ and I commutes with existential quantification as a left adjoint and with $\top$ by item (3).

It is easy to verify that composing with $I\colon \Sigma \to \Sigma$ gives a fibred functor, also denoted I, from p_J to p_I over $\mathcal{F}$. Likewise, composing with $J\colon \Sigma \to \Sigma$ gives a fibred functor, also denoted J, from p_I to p_J over $\mathcal{F}$.

Lemma 9.1.3. *Functor I is a fibred left adjoint to J and the triposes p_I and p_J are equivalent, as fibrations over I, via the functors I and J.*

Proof. Since both p_I and p_J are canonically presented, it suffices to consider the fibre over 1. Note first that $Ip \dashv\vdash IJp$ and that $Jp \dashv\vdash JIp$ in tripos p (by the modal logic for local maps, see (8.19) and (8.20)). Adjointness is shown as follows, for $p\colon \Sigma$ and $q\colon I\Sigma$,

$$\frac{\dfrac{\dfrac{\dfrac{p \vdash^{\mathsf{p}_J} Jq}{p \vdash^{\mathsf{p}} JJq}}{p \vdash^{\mathsf{p}} Jq}}{Ip \vdash^{\mathsf{p}} q}}{Ip \vdash^{\mathsf{p}_I} q}$$

For the equivalence, note first that $IJ \cong id$ because, for $q\colon I\Sigma$,

$$\frac{\dfrac{q \vdash^{\mathsf{p}_I} IJq}{q \vdash^{\mathsf{p}} IJq}}{q \vdash^{\mathsf{p}} Iq} \qquad \text{and} \qquad \frac{\dfrac{IJq \vdash^{\mathsf{p}_I} q}{IJq \vdash^{\mathsf{p}} q}}{Jq \vdash Jq}$$

(note that $q \vdash^{\mathsf{p}} Iq$ since $q\colon I\Sigma$, so $Iq = q$). Next note that $JI \cong id$ because, for $p\colon \Sigma$,

$$\frac{JIp \vdash^{\mathsf{p}_J} p}{JIp \vdash^{\mathsf{p}} Jp} \qquad \text{and} \qquad \frac{\dfrac{\dfrac{p \vdash^{\mathsf{p}_J} JIp}{p \vdash^{\mathsf{p}} JJIp}}{p \vdash^{\mathsf{p}} JIp}}{Ip \vdash Ip}$$

□

By the lemma it follows that p_I has all the first-order structure required in the definition of a tripos (since it is defined categorically and thus preserved by equivalence functors). It is clear that $id\colon I\Sigma \to I\Sigma$ is a generic object for p_I and thus p_I is indeed a tripos as claimed.

We now show that every local tripos gives rise to a localic local map of toposes.

Theorem 9.1.4. *Let* $\mathsf{p} = \mathcal{F}(-, \Sigma)$ *be a local* $\mathcal{F}$*-tripos qua* $I, J\colon \Sigma \to \Sigma$. *Then* p *gives rise to a localic local map of toposes from* $\mathcal{F}[\mathsf{p}]$ *to* $\mathcal{F}[\mathsf{p}_I]$.

Proof. Write $\mathbb{P}$ for the total category of p and $\mathbb{P}_I$ for the total category of p_I. We define three fibred functors over $\mathcal{F}$, as in

$$\begin{tikzcd} \mathbb{P}_I \arrow[rr, bend left, "\Delta"] \arrow[rr, bend right, "J"'] \arrow[dr, "\mathsf{p}_I"'] & & \mathbb{P} \arrow[ll, "I" description] \arrow[dl, "\mathsf{p}"] \\ & \mathcal{F} & \end{tikzcd} \qquad \Delta \dashv I \dashv J.$$

Functor Δ is simply the inclusion functor. Functor I is induced by composing with $I\colon \Sigma \to \Sigma$ and functor J is induced by composing with $J\colon \Sigma \to \Sigma$. It is easy to see that all three functors are fibred, since p and p_I are canonically presented, and that Δ is left adjoint to I and that I is left adjoint to J. The functor Δ is left exact since $I\Sigma$ is closed under finite limits in Σ by items (3) and (4) in the definition of a local tripos. Hence (Δ, I) is a geometric morphism of triposes, as is also (I, J). Functor Δ is clearly full and faithful and thus, by [MM92, Lemma 1, Section VII.4] we also have that J is full and faithful (it is also straightforward to verify directly that J is full and faithful).

It follows now, in the same way as in Section 6.2, that the geometric morphism from $\mathcal{F}[\mathsf{p}]$ to $\mathcal{F}[\mathsf{q}]$ induced by (Δ, I) is a localic local map. □

Conversely, we have:

Theorem 9.1.5. *Every localic local map of toposes arises from a local tripos (in the way given by the proof of Theorem 9.1.4).*

Proof. We use the notation from the introduction to this chapter and the introduction to this section. There we have already noted that $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ is a tripos. Call it p. Moreover, we also know that $\mathrm{D}_j\mathcal{E}[\mathsf{p}]$ is the topos $\mathcal{E}$.

Both the interior operation and the closure operation on $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ are natural, *i.e.*, commute with pullback. Thus they induce internal maps

$$I, J\colon \Gamma\Omega_{\mathcal{E}} \to \Gamma\Omega_{\mathcal{E}}$$

in $\mathrm{D}_j\mathcal{E}$ via an application of the Yoneda lemma to the natural isomorphisms in (9.1) such that I is internally left adjoint to J and I preserves finite limits. Moreover, the internal map I is deflationary and idempotent since the interior is so. Thus p is a local $\mathrm{D}_j\mathcal{E}$-tripos, canonically presented on $\Gamma\Omega_{\mathcal{E}}$.

Finally, it is clear that p_I is equivalent to the tripos $\begin{array}{c}\mathrm{Sub}(\mathrm{D}_j\mathcal{E})\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ since an open subobject of a discrete object is open. Hence $\mathrm{D}_j\mathcal{E}[\mathsf{p}_I]$ is nothing but $\mathrm{D}_j\mathcal{E}$ and the resulting local map from $\mathcal{E}$ to $\mathrm{D}_j\mathcal{E}$ is the one we started out with. $\square$

Remark 9.1.6. Let $\mathsf{p} = \mathcal{F}(-,\Sigma)$ be a local $\mathcal{F}$-tripos qua $I, J\colon \Sigma \to \Sigma$. From the equivalence of p_I and p_J and the description of implication and forall quantification in p_J (see Section 5.5), we get that the implication $\supset_{\mathsf{p}_I}$ and forall quantification $\forall^{\mathsf{p}_I}$ in p_I is given by

$$\varphi \supset_{\mathsf{p}_I} \psi = I(\varphi \supset_{\mathsf{p}} J\psi) \qquad \text{and} \qquad (\forall_F^{\mathsf{p}_I})(\varphi) = I(\forall_F(J\varphi)). \tag{9.2}$$

It is not hard to show that, in fact, we can leave out the application of J in (9.2), *i.e.*,

$$\varphi \supset_{\mathsf{p}_I} \psi \cong I(\varphi \supset_{\mathsf{p}} \psi) \qquad \text{and} \qquad (\forall_F^{\mathsf{p}_I})(\varphi) \cong I(\forall_F(\varphi)). \tag{9.3}$$

Hence the definitions of $\supset_{\mathsf{p}_I}$ and $\forall^{\mathsf{p}_I}$ are indeed as expected — in the case where p is the tripos $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$, the tripos p_I is equivalent to $\begin{array}{c}\mathrm{Sub}(\mathrm{D}_j\mathcal{E})\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ and the definition of implication and forall quantification here is exactly given as in (9.3) (see the definition of $\supset_d$ and $\forall_d$ in (8.3) on Page 183).

Example 9.1.7. The relative realizability tripos $\begin{array}{c}\mathbf{UFam}(A,A_\sharp)\\ \downarrow\mathsf{r}\\ \mathbf{Set}\end{array}$ from 5.1.4, see also Chapter 6, is a local tripos. The maps I and J are given by $\Delta\Gamma$ and $\nabla\Gamma$ respectively, see Section 6.2.

In this realizability example, the topos $\mathcal{E}$ is the topos $\mathrm{RT}(A, A_\sharp)$ and the topos $\mathrm{D}_j\mathcal{E}$ is the topos $\mathrm{RT}(A, A_\sharp)$ (see Chapter 10 for more on how our abstract theory of local maps relate concretely to the relative realizability model). Note that if we were just given the topos $\mathrm{RT}(A_\sharp)$ with the internal locale $\Gamma\Omega_{\mathrm{RT}(A,A_\sharp)}$ it would be quite hard to recognize the canonical tripos on this locale as being local, because it is complicated to calculate with internal adjoints *etc.* in $\mathrm{RT}(A_\sharp)$. That is one reason why it can be advantageous to describe a localic local map of toposes via a local tripos (rather than just in terms of internal locale theory).

Example 9.1.8 (Extensional Realizability). Let A be a PCA and let p be the standard **Set**-realizability tripos over A. Let $\mathbf{PER}(A)$ denote the category of partial equivalence relations over A. Define a tripos r over **Set** by taking predicates over sets I to be elements of $\mathbf{PER}(A)^I$, that is, I-indexed families of PER's. For two such families φ and ψ, we define the ordering over I to be $\varphi \vdash \psi$ iff there is an $a \in A$ such that, for all $i \in I$, a is in the domain of the PER $\psi(i)^{\varphi(i)}$ (the exponential in the category of PER's). See [Pit81, Section 1.6] and [vO97a] for more details. Then the tripos r is local over p, since the forgetful functor mapping a PER to its domain has both left and right (fibred) full and faithful adjoints. Over 1, the left adjoint maps a subset of A to the discrete PER on the subset and the right adjoint maps a subset of A to the PER with only one equivalence class. See [Pit81, Example 4.9(iii)] and [vO97a] for more details. We denote the topos resulting from the tripos r by $\mathrm{Ext}(A)$.

This example is special in the sense that the inclusion of $\mathrm{RT}(A)$ into $\mathrm{Ext}(A)$ is an *open* inclusion by Proposition 3.6 of [vO97a] (see [Joh77, Section 3.5] for more on open inclusions). That means that the principal topology j in $\mathrm{Ext}(A)$, for which $\mathrm{RT}(A)$ is equivalent to the category of sheaves, is of the form $j = (u \supset -)$ for some $u\colon 1 \to \Omega$. As a consequence, j has an internal left adjoint, namely $(u \times -)\colon \Omega \to \Omega$. This left adjoint induces the interior operation in $\mathrm{Ext}(A)$, so the interior operation *does* commute with pullback in this example.

9.2 Local Internal Locales

In this section we assume given a localic local map of toposes as in the introduction to this chapter. As explained in the previous section (see in particular the proof of Theorem 9.1.5), the interior operation and the closure operation in the resulting internal modal logic $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ correspond to internal maps I, J on the internal locale $\Gamma\Omega_{\mathcal{E}}$. In this section we employ some abstract theory relating internal locales and localic toposes to conclude that the internal maps have a very simple internal description. Moreover, we observe that a substantial part of the modal logic results *whenever* one has has an internal locale in a topos.

Convention 9.2.1. For brevity, we shall sometimes denote the topos $\mathrm{D}_j\mathcal{E}$ of discrete objects simply by $\mathcal{F}$. Also, we denote the internal locale $\Gamma\Omega_{\mathcal{E}}$ simply by Λ. (Thus $\Lambda \in \mathcal{F}$.)

Let $\mathbf{LTop}/\mathcal{F}$ denote the 2-category of localic $\mathcal{F}$-toposes and let $\mathbf{Locales}(\mathcal{F})$ denote the 2-category of internal locales in $\mathcal{F}$ and internal locale morphisms. The latter is defined in the standard way as the opposite of the category $\mathbf{Frames}(\mathcal{F})$ of internal frames in $\mathcal{F}$ and internal frame homomorphisms. See, *e.g.*, [MM92, Chapter IX] for a treatment of these categories in the case where $\mathcal{F} = \mathbf{Set}$. See, *e.g.*, [Joh79a] and [JT84] for more on internal locale theory.

Recall that the 2-category $\mathbf{LTop}/\mathcal{F}$ is equivalent to $\mathbf{Locales}(\mathcal{F})$ [Joh79a, Theorem 2.7]. The topos $\mathcal{F}$ is the terminal object in $\mathbf{LTop}/\mathcal{F}$ and the subobject classifier $\Omega = \Omega_{\mathcal{F}}$ of $\mathcal{F}$ is the terminal object in $\mathbf{Locales}(\mathcal{F})$ (see [Joh79a] for a proof of this fact). By the equivalence of $\mathbf{LTop}/\mathcal{F}$ and $\mathbf{Locales}(\mathcal{F})$, the unique geometric morphism $(\Delta, \Gamma)\colon \mathcal{E} \to \mathcal{F}$ in $\mathbf{LTop}/\mathcal{F}$ corresponds to the unique internal locale map from Λ to Ω. We also denote this unique locale map by (Δ, Γ). Thus $\Gamma\colon \Lambda \to \Omega$ and $\Delta\colon \Omega \to \Lambda$ are maps in $\mathcal{F}$, with Δ internally left adjoint to Γ, and Δ left exact, as depicted in:

$$\Delta \text{ lex} \qquad \text{and} \qquad \begin{tikzcd} \Lambda \arrow[r, shift left, "\Gamma"] \arrow[r, phantom, "\top"] & \Omega \arrow[l, shift left, "\Delta"] \end{tikzcd} \qquad \text{in } \mathcal{F}.$$

Likewise the geometric morphism $(\Gamma, \nabla)\colon \mathcal{F} \to \mathcal{E}$, which is a point of $\mathcal{E}$ in $\mathbf{LTop}/\mathcal{F}$, because

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "{(\Gamma,\nabla)}"] \arrow[dr, "{(id,id)}"'] & & \mathcal{E} \arrow[dl, "{(\Delta,\Gamma)}"] \\ & \mathcal{F} & \end{tikzcd}$$

commutes in the 2-category of toposes and geometric morphisms (*i.e.*, $\Gamma\nabla \cong id$), corresponds to a point of Λ in $\mathbf{Locales}(\mathcal{F})$, that is, a map from the terminal object Ω to Λ in $\mathbf{Locales}(\mathcal{F})$. We denote this point by (Γ, ∇), so $\nabla\colon \Omega \to \Lambda$ in $\mathcal{F}$. It also follows by the equivalence of $\mathbf{LTop}/\mathcal{F}$ and $\mathbf{Locales}(\mathcal{F})$ that $\Gamma\Delta = id_\Omega$ and $\Gamma\nabla = id_\Omega$ (see Lemma 2.8 and the bottom of Page II.18 in [Joh79a] for more details). Summarizing we have the following diagram in $\mathcal{F}$:

$$\begin{tikzcd} \Lambda \arrow[d, "\Gamma"] \\ \Omega \arrow[u, bend left=60, "\Delta"] \arrow[u, bend right=60, "\nabla"'] \end{tikzcd} \qquad \begin{array}{lll} \Delta \dashv \Gamma \dashv \nabla & \text{and} & \Delta \text{ lex} \\ \Gamma\Delta = id_\Omega & \text{and} & \Gamma\nabla = id_\Omega \end{array} \qquad \text{in } \mathcal{F} \tag{9.4}$$

We call an arbitrary internal locale Λ in an arbitrary topos $\mathcal{F}$ satisfying the conditions set out in (9.4) a **local internal locale**.

Since the interior operation in $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ is simply the functor $\Delta\Gamma$, see Proposition 8.3.3, we have that the corresponding internal map I on Λ is the internal functor $I = \Delta\Gamma\colon \Lambda \to \Lambda$. The internal map $J\colon \Lambda \to \Lambda$ corresponding to the closure operation in $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ is determined (by uniqueness of adjoints) by it being the right adjoint to I. By composing adjoints we see that $\nabla\Gamma\colon \Lambda \to \Lambda$ is right adjoint to $I = \Delta\Gamma$ and thus $J = \nabla\Gamma$.

Hence by abstract reasoning we have found out what the internal maps I and J on Λ are. We remark that it is also possible to give a more down-to-earth (but longer) derivation of what I and J are by employing (1) the fact that I and J are obtained via the Yoneda lemma (see the proof of Theorem 9.1.5) and (2) results of Johnstone [Joh79a] and Mikkelsen [Mik76] concerning the unique internal map to the terminal locale.

We denote the top element in Λ by 1_Λ and we write $=_\Lambda$ (or simply $=$) for the equality on Λ. By [Joh79a] we have that Δ and Γ are internally given as

$$\begin{aligned}\Delta(p) &= \bigvee\{\, 1_\Lambda \mid p \,\},\\ \Gamma(x) &= (x =_\Lambda 1_\Lambda),\end{aligned} \tag{9.5}$$

where $\bigvee$ is of course the sup in Λ. The set $\{\, 1_\Lambda \mid p \,\}$ is written using an abuse of notation; more properly we should write $\{\, x\colon \Lambda \mid x = 1_\Lambda \wedge p \,\}$. Since ∇ is right adjoint to Γ we get in the usual way (see, *e.g.*, [MM92, Proof of Lemma IX.1.1, Page 474]) that ∇ is given by

$$\begin{aligned}\nabla(p) &= \bigvee\{\, x \mid \Gamma x \le p \,\}\\ &= \bigvee\{\, x \mid (x = 1_\Lambda) \le p \,\},\end{aligned}$$

where $\le$ is the ordering on Ω (*i.e.*, implication $\supset$).

Hence I and J are given by

$$\begin{aligned}I(x) = \Delta\Gamma(x) &= \bigvee\{\, 1_\Lambda \mid x = 1_\Lambda \,\},\\ J(x) = \nabla\Gamma(x) &= \bigvee\{\, y \mid \bigvee\{\, 1_\Lambda \mid y = 1_\Lambda \,\} \le x \,\}\\ &= \bigvee\{\, y \mid y = 1_\Lambda \supset x = 1_\Lambda \,\}.\end{aligned}$$

Example 9.2.2. Recall Example 7.1.3(i), where X is a topological space with a generic point x. Then the base topos $\mathcal{F}$ is the category **Set**, Ω is the 2-element set $\{0, 1\}$, and $\Lambda = \mathcal{O}(X)$ is the locale of open sets, where the ordering is inclusion and for which $\bigvee$ is union of open sets. Using the above formulas for Δ, Γ, and ∇ we find that $\Delta\colon 2 \to \mathcal{O}(X)$ is given by

$$\Delta(0) = \emptyset \qquad \text{and} \qquad \Delta(1) = X,$$

the map $\Gamma\colon \mathcal{O}(X) \to 2$ is given by

$$\Gamma(U) = \begin{cases} 1 & \text{if } U = X, \\ 0 & \text{otherwise,} \end{cases}$$

and ∇ is given by

$$\nabla(0) = \bigcup_{x \notin U, U \in \mathcal{O}(X)} U \qquad \text{and} \qquad \nabla(1) = X.$$

Thus $I, J\colon \mathcal{O}(X) \to \mathcal{O}(X)$ are given by

$$I(U) = \begin{cases} X & \text{if } U = X, \\ \emptyset & \text{otherwise,} \end{cases}$$
$$J(U) = \begin{cases} X & \text{if } U = X, \\ \bigcup_{x \notin V, V \in \mathcal{O}(X)} V & \text{otherwise.} \end{cases}$$

If X is a Scott domain (and the generic point $x \in X$ thus is the bottom element $\bot \in X$), then $J(U) = X \setminus \{\bot\}$, whenever $U \neq X$.

9.2.1 Some Remarks on the Existence of ∇

Let $\mathcal{F}$ be an arbitrary topos with subobject classifier Ω and an internal locale Λ. Denote the unique locale map from Λ to Ω by (Δ, Γ) as in:

$$\Delta \text{ lex} \qquad \text{and} \qquad \begin{tikzcd} \Lambda \arrow[r, shift left, "\Gamma"] \arrow[r, phantom, "\top"] & \Omega \arrow[l, shift left, "\Delta"] \end{tikzcd} \qquad \text{in } \mathcal{F},$$

where Δ and Γ are given as in (9.5).

By (the internal) adjoint functor theorem, Γ has a right adjoint ∇ iff Γ

preserves all colimits, that is, iff internally in $\mathcal{F}$ we have

$$\Gamma(\bigvee_{i\in I} x_i) =_\Omega \bigvee_{i\in I} \Gamma x_i \tag{9.6}$$

$$\begin{aligned}
&\iff \Big((\bigvee_{i\in I} x_i) = 1_\Lambda\Big) \leq_\Omega \Big(\bigvee_{i\in I}(x_i = 1_\Lambda)\Big) \\
&\iff \Big((\bigvee_{i\in I} x_i) = 1_\Lambda\Big) \supset \Big(\bigvee_{i\in I}(x_i = 1_\Lambda)\Big) \\
&\iff \Big((\bigvee_{i\in I} x_i) = 1_\Lambda\Big) \supset \exists i \colon I.\ (x_i = 1_\Lambda).
\end{aligned} \tag{9.7}$$

If Γ satisfies condition (9.6) or, equivalently, condition (9.7), we say that Γ is **additive**.

Lemma 9.2.3. *Suppose that $\Gamma \colon \Lambda \to \Omega$ is additive (and thus has a right adjoint ∇). Then Γ is epic.*

Proof. Γ is epic iff

$$\forall p \colon \Omega.\ \exists x \colon \Lambda.\ \Gamma x =_\Omega p,$$

holds in the internal logic of $\mathcal{F}$, that is, iff

$$\forall p \colon \Omega.\ \exists x \colon \Lambda.\ (x = 1_\Lambda) \supset\subset p \tag{9.8}$$

holds in the internal logic. We prove (9.8) by arguing internally. Let p be an arbitrary element of Ω and take x to be $\Delta p = \bigvee\{\,1_\Lambda \mid p\,\}$. We are then to show that

$$\Big(\bigvee\{\,1_\Lambda \mid p\,\} = 1_\Lambda\Big) \supset\subset p.$$

Suppose first that p holds. Then $1_\Lambda \in \{\,1_\Lambda \mid p\,\}$ so clearly $\bigvee\{\,1_\Lambda \mid p\,\} = 1_\Lambda$. For the other direction, suppose that

$$\bigvee\{\,1_\Lambda \mid p\,\} = 1_\Lambda$$

holds. The set $\{\,1_\Lambda \mid p\,\}$ is short for $\{\,x \colon \Lambda \mid x = 1_\Lambda \wedge p\,\}$. By additivity of Γ, see (9.7), there exists an x in $\{\,x \colon \Lambda \mid x = 1_\Lambda \wedge p\,\}$ such that $x = 1_\Lambda$. Hence there exists an x such that $x = 1_\Lambda \wedge p$. Thus p holds, as required. □

Corollary 9.2.4. *Suppose that $\Gamma \colon \Lambda \to \Omega$ is additive (and thus has a right adjoint ∇). Then Δ is monic and $\Gamma\Delta = id_\Omega$.*

Proof. By Lemma 2.8 in [Joh79a] and Lemma 9.2.3 above. □

By Corollary 9.2.4 the following proposition follows.

Proposition 9.2.5. *Let Λ be an internal locale in a topos $\mathcal{F}$ and let (Δ, Γ) denote the unique map to the terminal locale Ω in $\mathcal{F}$. Then Λ is local iff $\Gamma\colon \Lambda \to \Omega$ is additive (*i.e., *iff equation* (9.7) *holds).*

This proposition is a special case of Proposition 1.7 of Johnstone and Moerdijk [JM89]. Johnstone and Moerdijk prove that an $\mathcal{F}$-topos $\mathcal{E}$ is local iff there exists an internal local site for $\mathcal{E}$ in $\mathcal{F}$, where a site D is local if it's underlying category has a terminal object t and, moreover, it is internally valid in $\mathcal{F}$ that, whenever $(d_i \to t)_{i \in I}$ is a cover in D, there exists an $i\colon I$ such that $d_i \to t$ has a section. Taking the site to be the internal locale Λ with the usual sup-topology, we see that the condition of Johnstone and Moerdijk is exactly our condition (9.7).

Example 9.2.6. Continuing Example 9.2.2, note that the additivity condition in (9.7) says that whenever we have an open cover of X, $\bigcup_{i \in I} U_i = X$, then there exists an $i \in I$ such that the open set U_i already covers X. That this holds is clear since the generic point $x \in X$ must be in one of the U_i's and the only open set containing x is X itself, so U_i must equal X.

Remark 9.2.7. It is interesting to note that *for any* internal locale Λ in any topos $\mathcal{F}$ (*i.e.*, even if Λ is not local), the induced interior map $I = \Delta\Gamma\colon \Lambda \to \Lambda$ satisfies the axioms and rules for the box operator in the propositional modal logic S4: for all $x, y\colon \Lambda$,

$$I(x \supset_\Lambda y) \vdash_\Lambda Ix \supset_\Lambda Iy$$
$$Ix \vdash_\Lambda x$$
$$Ix \vdash_\Lambda IIx$$
$$\frac{\vdash_\Lambda x}{\vdash_\Lambda Ix}$$

($\vdash_\Lambda$ is of course the ordering $\leq_\Lambda$ on Λ). This can be proved directly using the internal adjoints Δ and Γ. However, it also follows directly by results of Biermann and de Paiva [BdP96]: Let

$$(I, \epsilon, \delta)$$

be the comonad on Λ induced by $\Delta \dashv \Gamma$. Thus

$$I = \Delta\Gamma$$

and

$$\epsilon\colon \Delta\Gamma \Rightarrow id$$

is the counit of $\Delta \dashv \Gamma$ and

$$\delta = \Delta\eta\Gamma = id$$

with $\eta\colon id \Rightarrow \Gamma\Delta$ the unit of $\Delta \dashv \Gamma$, which is an identity. Then (I, ϵ, δ) is in fact a left exact (and thus suitably monoidal) comonad on Λ. Moreover, Λ is a (internal) cartesian closed category with coproducts. Therefore we have a model of intuitionistic propositional S4 modal logic [BdP96].

Chapter 10

More on the Relative Realizability Topos $\mathrm{RT}(A, A_\sharp)$

We now return to consider the relative realizability topos $\mathrm{RT}(A, A_\sharp)$ from Chapter 6 in the light gained from our analysis of local maps of toposes in Chapters 7–9. That is the main point of this chapter. However, we also use this chapter to collect some other specific material on $\mathrm{RT}(A, A_\sharp)$. Most of this material is obtained by verifying that known results for standard realizability toposes can be carried over to the relative realizability setting.

Let us now outline the contents of the chapter in more detail.

In Section 10.1 we explicitly characterize some of the objects and operations used in the abstract development in Chapters 7–9. We argue that the interior operation can be seen intuitively to carve out the subset of computable elements of an object. We also include a concrete treatment of the associated discrete object functor.

In Section 10.2 we describe explicitly how the modal logic for local maps is interpreted in $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A, A_\sharp)$.

In Section 10.3 we comment on the relationship between the logics of $\mathrm{RT}(A, A_\sharp)$ and $\mathrm{RT}(A_\sharp)$; in particular we prove that the inclusion of $\mathrm{RT}(A_\sharp)$ into $\mathrm{RT}(A, A_\sharp)$ does not preserve all of first-order logic.

In Section 10.5 we present a number of facts concerning the double-negation topology in $\mathrm{RT}(A, A_\sharp)$. Many of the properties concerning the double-negation topology in standard realizability toposes also hold for the relative realizability topos $\mathrm{RT}(A, A_\sharp)$. We also define subcategories of assemblies and modest sets over A with respect to $A_\sharp$ and show that they have the same relationships with each other and with $\mathrm{RT}(A, A_\sharp)$ as the corresponding categories of assemblies and modest sets have for standard realizability toposes. Furthermore, we show that the interior operation has a particularly simple description for $\neg\neg$-separated objects and we present a concrete example in which the interior operation indeed precisely returns the subset of computable elements of an object.

In Section 10.6 we show that $\mathrm{RT}(A, A_\sharp)$ can also be described as the exact completion of it's full subcategory of projectives in the same way as $\mathrm{RT}(A)$ can.

10.1 Some Objects and Maps in $\mathrm{RT}(A, A_\sharp)$

Recall that by Theorem 6.2.3, $(\Delta, \Gamma)\colon \mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A_\sharp)$ is a localic local map of toposes. Thus by the discussion in Section 7.2, we have the following picture:

$$
\begin{tikzcd}
& \mathrm{RT}(A, A_\sharp) \arrow[dl, "\Gamma"] \arrow[dr, "\mathbf{a}"] & \\
\mathrm{RT}(A_\sharp) \arrow[ur, bend left, "\Delta"] \arrow[ur, bend right, "\nabla"'] \arrow[rr, shift left, "\mathbf{a}\nabla"] & & \mathrm{Sh}_j\mathrm{RT}(A, A_\sharp) \arrow[ul, bend right, "i"'] \arrow[ul, bend left, "\Delta\Gamma i"] \arrow[ll, shift left, "\Gamma i"]
\end{tikzcd}
$$

$$
\begin{aligned}
&\Delta \dashv \Gamma \dashv \nabla, \\
&\Delta\Gamma i \dashv \mathbf{a} \dashv i, \\
&\Gamma i\,\mathbf{a} \cong \Gamma, \\
&\mathbf{a}\,\nabla\Gamma i \cong id, \\
&\Gamma i\,\mathbf{a}\,\nabla \cong id, \\
&\Delta \text{ f+f, lex,} \\
&\nabla \text{ f+f.}
\end{aligned}
$$

In this section we identify the topology j in $\mathrm{RT}(A, A_\sharp)$. To this end we first explicate what the subobject classifiers are in $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A, A_\sharp)$ (Subsection 10.1.1). Then we identify the topology j (Subsection 10.1.2) and we describe what the discrete objects are in $\mathrm{RT}(A, A_\sharp)$ (Subsection 10.1.3). Moreover, we show how to calculate the action of the interior operation (Subsection 10.1.4), and we give a concrete treatment in the $\mathrm{RT}(A, A_\sharp)$ model of the associated discrete object functor (Subsection 10.1.5).

10.1.1 Subobject Classifiers in $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A, A_\sharp)$

By Section 5.2.3, the subobject classifier $\Omega_{\mathrm{RT}(A,A_\sharp)}$ in $\mathrm{RT}(A, A_\sharp)$ is given by

$$\Omega_{\mathrm{RT}(A,A_\sharp)} = (PA, \approx)$$

with

$$p, q\colon PA \mid p \approx q \stackrel{\text{def}}{=} p \supset\!\subset q = \{\, \langle a, b\rangle \mid a\colon p \supset q \text{ and } b\colon q \supset p \text{ and } a, b \in A \,\}.$$

Note that the biimplication is given as in the tripos $\begin{array}{c}\mathbf{UFam}(A,A_\sharp)\\ \downarrow r\\ \mathbf{Set}\end{array}$ underlying $\mathrm{RT}(A, A_\sharp)$. We simply write $\Omega_{\mathrm{RT}(A,A_\sharp)} = (PA, \supset\!\subset)$, and sometimes, when no confusion can arise, we leave out the subscript on $\Omega_{\mathrm{RT}(A,A_\sharp)}$.

The subobject classifier, $\Omega_{\mathrm{RT}(A_\sharp)}$ in $\mathrm{RT}(A_\sharp)$ is given by

$$\Omega_{\mathrm{RT}(A,A_\sharp)} = (PA_\sharp, \approx)$$

with

$$p, q\colon PA_\sharp \mid p \approx q \stackrel{\text{def}}{=} p \supset\!\subset q = \{\, \langle a, b\rangle \mid a\colon p \supset q \text{ and } b\colon q \supset p \text{ and } a, b \in A_\sharp \,\}.$$

Note that the biimplication is given as in the tripos $\begin{array}{c}\mathbf{UFam}(A_\sharp)\\ \downarrow q\\ \mathbf{Set}\end{array}$ underlying $\mathrm{RT}(A_\sharp)$, that is, it involves computable realizers from $A_\sharp$. We simply write $\Omega_{\mathrm{RT}(A_\sharp)} = (PA_\sharp, \supset\!\subset^\sharp)$, and sometimes, when no confusion can arise, we leave out the subscript on $\Omega_{\mathrm{RT}(A_\sharp)}$.

10.1.2 The topology j in $\mathrm{RT}(A, A_\sharp)$

Let $\Omega = \Omega_{\mathrm{RT}(A,A_\sharp)}$ be the subobject classifier in $\mathrm{RT}(A, A_\sharp)$. By Section 5.5, the Lawvere-Tierney topology $j\colon \Omega \to \Omega$ classifies the strict predicate J on Ω given by

$$J(p) \stackrel{\text{def}}{=} \nabla\Gamma(p) = \bigcup_{\varphi \in PA} \Big(\varphi \wedge (\varphi \cap A_\sharp \supset p \cap A_\sharp)\Big).$$

Thus by Section 5.2.3 the classifying map $j\colon \Omega \to \Omega$ is represented by the functional relation

$$\begin{aligned} p, q\colon \Omega \mid j(p,q) &\stackrel{\text{def}}{=} (p \supset\subset p) \wedge J(p) \supset\subset q \\ &\cong J(p) \supset\subset q. \end{aligned}$$

10.1.3 Discrete Objects in $\mathrm{RT}(A, A_\sharp)$

Since the functor $\Delta\colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ really is an identity functor, an object in $\mathrm{RT}(A, A_\sharp)$ is discrete iff it is isomorphic to an object from $\mathrm{RT}(A, A_\sharp)$. In other words, the discrete objects in $\mathrm{RT}(A, A_\sharp)$ are the replete image of $\mathrm{RT}(A_\sharp)$ in $\mathrm{RT}(A, A_\sharp)$.

10.1.4 Interior in $\mathrm{RT}(A, A_\sharp)$

By the proof of Proposition 7.3.39, the interior of an object $X \in \mathrm{RT}(A, A_\sharp)$ is calculated as the image of the counit $\epsilon_X : \Delta\Gamma i\,\mathbf{a} X \to X$ of $\Delta\Gamma i \dashv \mathbf{a}$. Since $\Gamma i\,\mathbf{a} \cong \Gamma$, the is equivalent to taking the image of the counit $\epsilon_X : \Delta\Gamma X \to X$ of $\Delta \dashv \Gamma$.

Let $(X, \approx_X)$ be an object of $\mathrm{RT}(A, A_\sharp)$. As explained on Page 132 the counit at $(X, \approx_X)$ of $\Delta \dashv \Gamma$

$$\epsilon_{(X,\approx_X)}\colon \Delta\Gamma(X, \approx_X) \to (X, \approx_X)$$

is represented by the functional relation

$$E(x, x') = (x \approx_X x) \cap A_\sharp \wedge (x \approx_X x').$$

Thus the interior of $(X, \approx_X)$ (the image of $\epsilon_{(X,\approx)}$) is

$$(X, \approx_X)^\circ = (X, \approx'),$$

where

$$x, x'\colon X \mid x \approx'_X x' \stackrel{\text{def}}{=} x \approx_X x' \wedge \exists x_0\colon X.\ (x_0 \approx_X x_0) \cap A_\sharp \wedge (x_0 \approx_X x).$$

Remark 10.1.1. Let us attempt to give an intuitive reading of the interior operation. We think of an object $(X, \approx_X)$ in $\mathrm{RT}(A, A_\sharp)$ as a set X with a partial equivalence relation $\approx_X$ on it. Thus "elements" of $(X, \approx_X)$ are really to be thought of as equivalence classes of X w.r.t. $\approx_X$. The partial equivalence relation is given via continuous realizers, in the sense that $x \approx_X x'$ is a subset of A. We can say that an $x_0 \in X$ is "computable" if $(x_0 \approx_X x_0) \cap A_\sharp \neq \emptyset$ and extend this to say that an element of $(X, \approx)$, *i.e.*, an equivalence class, is "computable" if the equivalence class contains a computable representative. Thinking in this way, we read the interior of an object $(X, \approx)$ as the subset of $(X, \approx)$ consisting of all the computable elements (those equivalence classes that have a computable representative x_0). In Example 10.5.19 we give a concrete example where it is absolutely clear that the interior really does give the computable elements.

Continuing in this intuitive style, we also note the difference between $\Delta\Gamma(X, \approx_X)$ and the interior $(X, \approx_X)^\circ$. The interior $(X, \approx_X)^\circ$ is a quotient of $\Delta\Gamma(X, \approx_X)$. indeed we can think of $\Delta\Gamma(X, \approx_X)$ as a refinement of $(X, \approx_X)$ in which the equivalence classes of $(X, \approx_X)$ are possibly split up into several equivalence classes since the partial equivalence relation $\approx_X \cap A_\sharp$ is finer (makes more distinctions) than the partial equivalence relation $\approx_X$. The idea is that computably we cannot make as many identifications as we can continuously.

Proposition 10.1.2. *The subobject classifier Ω in $\mathrm{RT}(A, A_\sharp)$ is open.*

Proof. The interior of Ω is $\Omega^\circ = (PA, \approx')$, where

$$p, q \colon PA \mid p \approx' q \stackrel{\text{def}}{=} p \supset\!\subset q \wedge \exists r \colon PA.\ (r \supset\!\subset r) \cap A_\sharp \wedge (r \supset\!\subset p).$$

But clearly $\approx'$ is isomorphic to $\supset\!\subset$ in $\mathbf{UFam}(A, A_\sharp)_{PA \times PA}$, that is,

$$p, q \colon PA \mid p \approx' q \dashv\vdash p \supset\!\subset q$$

holds in the internal logic of the tripos $\mathbf{q}$ underlying $\mathrm{RT}(A, A_\sharp)$. The reason is that we can just take r to be p — note that $(r \supset\!\subset r) \cap A_\sharp$ is trivially realized by the identity function. Thus $\Omega^\circ \cong \Omega$. □

The fact that Ω is open can be used to show that the open objects are not closed under finite limits in $\mathrm{RT}(A, A_\sharp)$:

Proposition 10.1.3. *The open objects in $\mathrm{RT}(A, A_\sharp)$ are not closed under finite limits in $\mathrm{RT}(A, A_\sharp)$.*

Proof. Suppose for a contradiction that the open objects are closed under finite limits. Let X be an open object and let $U \rightarrowtail X$ be any subobject of X. Consider the following pullback diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ X \arrow[r, "\chi_U"'] & \Omega_{\mathrm{RT}(A,A_\sharp)} \end{tikzcd}$$

with χ_U the classifying map of $U \rightarrowtail X$. Then by the assumption that the open objects are closed under finite limits we get that U is open, since X, 1 and Ω (by the previous proposition) are all open. Thus we conclude that any subobject U of an open object X is open, which is clearly not the case (think of $X = 1$), and hence we have a contradiction. $\square$

Remark 10.1.4. From the above proposition we see that in general we cannot expect the interior operator to commute with pullbacks along open objects. Therefore we cannot in general enlarge the collection of types for the modal logic for local maps from the discrete objects to the open objects.

10.1.5 The Associated Discrete Object Functor in $\mathrm{RT}(A, A_\sharp)$

The functor Γ is of course the associated discrete object functor. For explanatory purposes we now work out the abstract construction of the associated discrete object functor (see the proof of Theorem 7.3.31) in $\mathrm{RT}(A, A_\sharp)$. The hope is that this will give the reader a more intuitive understanding of the abstract construction of the associated discrete object functor.

Let $(X, \approx_X)$ be an object of $\mathrm{RT}(A, A_\sharp)$. Consider the diagram for the associated discrete object functor's action on $(X, \approx_X)$:

$$\begin{tikzcd} K_{e^\circ}{}^\circ \arrow[r, tail] & K_{e^\circ} \arrow[d, shift left] \arrow[d, shift right] & & \\ & (X, E'_X) \arrow[r, tail, "m"] \arrow[dl] \arrow[d, "e^\circ"] & (X, E_X) \arrow[r, tail] \arrow[d, "e"] & (X, \exists_{\delta_X}(T)) \\ (X, \widetilde{\approx_X})^\circ \arrow[r, two heads, "h"'] & (X, \approx'_X) \arrow[r, tail] & (X, \approx_X), & \end{tikzcd}$$

where

- $(X, \exists_{\delta_X}(T))$ is an object of $\mathrm{RT}(A_\sharp)$ (*i.e.*, is discrete), where $\exists_{\delta_X}(T)$ is the constant equality:

$$\exists_{\delta_X}(T)(x, x') = \begin{cases} A & \text{if } x = x', \\ \emptyset & \text{otherwise.} \end{cases}$$

- E_X is given as

$$E_X(x, x') = \begin{cases} x \approx_X x' & \text{if } x = x', \\ \emptyset & \text{otherwise.} \end{cases}$$

- e is represented by the (functional) relation $\approx_X$.
- $(X, \approx'_X)$ is the interior of X (see above).
- (X, E'_X) is the interior of X, with E'_X given as

$$E_X(x, x') = \begin{cases} (x \approx_X x') \cap A_\sharp & \text{if } x = x' \text{ and } (x \approx_X x') \cap A_\sharp \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

  Note that (X, E'_X) is an object of $\mathrm{RT}(A_\sharp)$ since the equality predicate is $PA_\sharp$-valued — this phenomenon is an instance of the fact that an open subobject of a discrete object, in this case $(X, \exists_{\delta_X}(T))$, is again discrete.

- e_0 is represented by the functional relation E_0 given by

$$E_0(x, x') = E'_X(x, x) \wedge x \approx_X x'.$$

- K_{e° is represented by a strict predicate on $(X, E'_X) \times (X, E'_X)$ given by

$$K_{e^\circ}(x, x') = E'_X(x) \wedge E'_X(x') \wedge x \approx_X x'.$$

- $K_{e^\circ}{}^\circ$ is represented by a strict predicate on $(X, E'_X) \times (X, E'_X)$ given by

$$\begin{aligned} K_{e^\circ}{}^\circ(x, x') &= E'_X(x) \wedge E'_X(x') \wedge (x \approx_X x') \cap A_\sharp \\ &\cong (x \approx_X x') \cap A_\sharp \end{aligned}$$

  (the isomorphism being in $\mathbf{UFam}(A, A_\sharp)_{X \times X}$).

- $(\widetilde{X, \approx_X})^\circ$ is therefore isomorphic to $\Gamma(X, \approx_X) = (X, \Gamma \approx_X)$, where, recall, $(\Gamma \approx_X)(x, x') = (x \approx_X x') \cap A_\sharp$.

Thus we have seen concretely how the associated discrete object functor applied to an object $(X, \approx_X)$ indeed gives $\Gamma(X, \approx_X)$.

It was partly via concrete calculations such as these that I found the abstract construction of the associated discrete object functor in the proof of Theorem 7.3.31.

10.2 Interpretation of the Modal Logic in $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A, A_\sharp)$

By the results in Chapter 9, the fibration $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow \\ \mathrm{RT}(A,A_\sharp)\end{smallmatrix}$ obtained by change-of-base as in

$$\begin{tikzcd}
\mathrm{Pred} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sub}(\mathrm{RT}(A, A_\sharp)) \arrow[d] \\
\mathrm{RT}(A_\sharp) \arrow[r, hook, "\Delta"'] & \mathrm{RT}(A, A_\sharp)
\end{tikzcd}$$

is a local tripos which is a model of the modal logic for local maps described in Section 8.3.

Thus in this model of the modal logic, types and terms are interpreted in the standard way as objects and morphisms in $\mathrm{RT}(A, A_\sharp)$. A context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ is interpreted in the standard way as an object $I = X_1 \times \cdots \times X_n$ in $\mathrm{RT}(A_\sharp)$, with X_i the object interpreting σ_i. Formulas $\Gamma \vdash \varphi \colon \mathsf{Prop}$ in context Γ are interpreted as subobjects in $\mathcal{E}$ of $\Delta I = I$ (with I the interpretation of Γ). As explained in Section 5.2.3, subobjects in $\mathrm{RT}(A, A_\sharp)$ are equivalent to strict and extensional predicates in the tripos $\begin{smallmatrix}\mathbf{UFam}(A,A_\sharp)\\ \downarrow\mathsf{r} \\ \mathbf{Set}\end{smallmatrix}$ underlying $\mathrm{RT}(A, A_\sharp)$. In other words, if $(I, \approx_I)$ is an object of $\mathrm{RT}(A_\sharp)$ (a context), then a predicate φ in $\mathrm{Pred}_{(I,\approx_I)}$ is a strict, extensional predicate $\varphi \colon I \to PA$; strict in the sense that

$$i \colon I \mid \varphi(i) \vdash i \approx_I i \tag{10.1}$$

holds in the logic of tripos r and extensional in the sense that

$$i, i' \colon I \mid (i \approx_I i') \wedge \varphi(i) \vdash \varphi(i') \tag{10.2}$$

holds in the logic of tripos r. We recall that (10.1) means that

$$\exists a \in A_\sharp.\ \forall i \in I.\ \forall b \in \varphi(i).\ a \cdot b \in (i \approx_I i)$$

and that (10.2) means that

$$\exists a \in A_\sharp.\ \forall i, i' \in I.\ \forall b \in (i \approx_I i').\ \forall c \in \varphi(i).\ a \cdot \langle b, c\rangle \in \varphi(i').$$

The atomic predicates $\top$, $\bot$, logical connectives $\wedge$, $\supset$, $\vee$, and quantifiers $\exists$ and $\forall$ are all interpreted in as in the logic of $\mathrm{RT}(A, A_\sharp)$ (see Section 5.2.5).

Consider a formula in context $\Gamma \vdash \varphi\colon \mathsf{Prop}$ with Γ interpreted by an object $(I, \approx_I) \in \mathrm{RT}(A_\sharp)$ and φ interpreted as a strict, extensional predicate $\varphi\colon I \to PA$ on $(I, \approx_I)$. By Propositions 8.3.4 and 8.3.3, the interpretation of the formula $\Gamma \vdash \sharp\varphi\colon \mathsf{Prop}$ is the strict predicate $\sharp\varphi$ given by

$$(\sharp\varphi)(i) = \varphi(i) \cap A_\sharp.$$

Likewise the interpretation of the formula $\Gamma \vdash \flat\varphi\colon \mathsf{Prop}$ is, by Proposition 8.3.4 given by the closure operation associated with the topology j. By Subsection 10.1.2 above, $\Gamma \vdash \flat\varphi\colon \mathsf{Prop}$ is thus explicitly interpreted as the strict predicate $\flat\varphi$ given by

$$(\flat\varphi)(i) = \bigcup_{q \in PA} \Big(q \wedge \big(q \cap A_\sharp \supset \varphi(i) \cap A_\sharp\big)\Big).$$

As explained in Section 9.1, the internal locale $\Lambda = \Gamma(\Omega_{\mathrm{RT}(A,A_\sharp)}) = (PA, \approx')$ with

$$\begin{aligned} p, q\colon PA \mid p \approx' q &\stackrel{\text{def}}{=} (p \supset\subset q) \cap A_\sharp \\ &= \{\, \langle a, b\rangle \in A_\sharp \mid \forall c \in p.\ a \cdot c \in q \text{ and } \forall c \in q.\ b \cdot c \in p \,\} \end{aligned}$$

is a generic object for the tripos $\begin{array}{c}\mathbf{Pred}\\ \downarrow\\ \mathrm{RT}(A,A_\sharp)\end{array}$.

Summarizing we have that, for a type (objects) $(J, \approx_J)$ in $\mathrm{RT}(A_\sharp)$ and for strict and extensional predicates $\varphi\colon J \to PA$ and $\psi\colon J \to PA$ the propositional operations are interpreted as follows (on the right hand side of the equations we use the operations of the tripos $\begin{array}{c}\mathbf{UFam}(A,A_\sharp)\\ \downarrow \mathsf{r}\\ \mathbf{Set}\end{array}$ — we recall what they are by also giving the explicit set-theoretic definitions in terms of realizers).

$$\begin{aligned}
\top(j) &= (j \approx_J j)\\
(\varphi \wedge \psi)(j) &= \varphi(j) \wedge \psi(j)\\
&= \{\, \langle a, b\rangle \mid a \in \varphi(j) \text{ and } b \in \psi(j)\,\}\\
\bot(j) &= \emptyset\\
(\varphi \vee \psi)(j) &= \varphi(j) \vee \psi(j)\\
&= \{\, \langle \mathbf{K}, a\rangle \mid a \in \varphi(i)\,\} \cup \{\, \langle \mathbf{KI}, b\rangle \mid b \in \varphi(i)\,\}\\
(\varphi \supset \psi)(j) &= (j \approx_J j) \wedge (\varphi(j) \supset \psi(j))\\
&= \{\, \langle a, b\rangle \mid a \in (j \approx_J j) \text{ and } b \in A \text{ and } \forall c \in \varphi(j).\ b \cdot c \in \psi(j)\,\}\\
(\sharp\varphi)(j) &= \varphi(i) \cap A_\sharp\\
(\flat\varphi)(j) &= \bigcup_{q \in PA} \Big(q \wedge (q \cap A_\sharp \supset \varphi(i) \cap A_\sharp)\Big).
\end{aligned}$$

Given a strict and extensional predicate φ on $(I, \approx_I) \times (J, \approx_J)$ we have that

$$\begin{aligned}
(\exists j\colon (J, \approx_J).\ \varphi)(i) &= \exists j\colon J.\ (j \approx_J j) \wedge \varphi(i, j)\\
&= \bigcup_{j \in J} \{\, \langle a, b\rangle \mid a \in (j \approx_J j) \text{ and } b \in \varphi(i, j)\,\}\\
(\forall j\colon (J, \approx_J).\ \varphi)(i) &= \forall j\colon J.\ (j \approx_J j) \supset \varphi(i)\\
&= \bigcap_{j \in J} \{\, a \in A \mid \forall b \in (j \approx_J j).\ a \cdot b \in \varphi(i)\,\},
\end{aligned}$$

Finally, a closed predicate $\varphi\colon 1 \to PA$ is *valid* iff it contains a realizer in $A_\sharp$, so a (open) predicate $\varphi\colon J \to PA$ on $(J, \approx_J)$ is valid iff $(\forall j\colon (J, \approx_J).\ \varphi)$ contains a realizer in $A_\sharp$.

10.3 On the Relationship Between the Logic of $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A, A_\sharp)$

The results in Chapter 8 concerning the preservation of the interpretation of stable formulas (Theorem 8.2.11) and the results concerning Church's Thesis all apply in our case of $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A, A_\sharp)$. It is natural to ask if in fact a larger fragment of logic is preserved by the inclusion Δ of the discrete objects $\mathrm{RT}(A_\sharp)$ into $\mathrm{RT}(A, A_\sharp)$. In this section we prove a negative result: we show that the geometric morphism $(\Delta, \Gamma)\colon \mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A_\sharp)$ is not open. Recall that if a geometric morphism is open, then the inverse image functor (the inclusion $\Delta\colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ in this case) preserves first-order logic. See [MM92, Section IX.6] and [Joh80] for more on open maps of toposes.

Theorem 10.3.1. *The geometric morphism* $(\Delta, \Gamma)\colon \mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A_\sharp)$ *is not open.*

Proof. Suppose for a contradiction that (Δ, Γ) is open so that $\Delta\colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ preserves first-order logic. Since the functor $\mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A)$, call it Q, from Section 6.1 is logical we then have that the composite functor

$$Q\Delta\colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A)$$

preserves first-order logic. Recall that Δ preserves the natural numbers object N in $\mathrm{RT}(A_\sharp)$ as an inverse image functor. Moreover, Q also preserves the natural numbers object in $\mathrm{RT}(A, A_\sharp)$ since Q is logical and thus preserves finite limits and finite colimits (this is enough to preserve the natural numbers object by Freyd's characterization of it, see [Joh77, Theorem 6.14]). Hence $Q\Delta$ preserves the natural numbers object of $\mathrm{RT}(A_\sharp)$ and thus $Q\Delta$ preserves first-order arithmetic.

Now let A be the graph model PCA $\mathbb{P}$ from Example 3.1.15 and let $A_\sharp$ be the r.e. sub-model RE of $\mathbb{P}$ from Example 3.1.16. It is well-known that $\mathrm{RT}(RE)$ satisfies all instances of the arithmetical form of Church's Thesis CT_0 (see Definition 8.3.7 in Section 8.3.3). So by applying $Q\Delta$ we also get that $\mathrm{RT}(\mathbb{P})$ satisfies all instances of CT_0. But since it is also known that the valid first-order arithmetical sentences in $\mathrm{RT}(\mathbb{P})$ are exactly all the classically true sentences we have a contradiction since CT_0 is not valid classically [TvD88, Section 4.3.3].

The idea of using the logical functor Q in this proof is due to Jaap van Oosten. $\square$

10.4 On the Relation to RT(A)

In Section 6.1 we explained that $\mathrm{RT}(A)$ is the filter-quotient of $\mathrm{RT}(A, A_\sharp)$ by the filter of subobjects of 1 consisting of all those subobjects of 1 corresponding to inhabited subsets of A. We now remark that this (external) filter of subobjects of 1 actually arises in the standard way [Joh77, Page 319] from the *internal filter* on the internal locale $\Lambda = \Gamma(\Omega_{\mathrm{RT}(A, A_\sharp)})$ given (internally) by

$$\mathcal{F} = \{\, x \in \Lambda \mid x \neq \bot_\Lambda \,\}.$$

(We sketch below how one proves this claim.) This means that *in the logic of* $\mathrm{RT}(A_\sharp)$ we can describe

- the construction of $\mathrm{RT}(A, A_\sharp)$: by the tripos-to-topos construction on the internal locale Λ
- the construction of $\mathrm{RT}(A)$: by the tripos-to-topos construction applied to the internal locale Λ but with entailment of $\varphi, \psi\colon X \to \Lambda$ redefined to mean $(\forall x\colon X.\ \varphi(x) \supset \psi(x)) \in \mathcal{F}$, where $\forall$ is interpreted by the internal meet of Λ (see Section 5.1.3)

Whether these observations are of any practical import remains to be seen since we can also describe all three realizability toposes in the logic of the tripos $\begin{array}{c}\mathbf{UFam}(A,A_\sharp)\\ \downarrow\mathsf{r}\\ \mathbf{Set}\end{array}$ which may be simpler to calculate with.

We now show that $\mathcal{F}$ is an internal filter. It of course suffices to show that

$$\forall x, y\colon \Lambda.\ (x \neq \bot_\Lambda) \wedge (y \neq \bot_\Lambda) \supset (x \wedge_\Lambda y) \neq \bot_\Lambda \tag{10.3}$$

is valid in the internal logic of $\mathrm{RT}(A_\sharp)$. The intuitive argument is that Λ is the internal poset reflection of the fibre over 1 in $\begin{array}{c}\mathbf{UFam}(A,A_\sharp)\\ \downarrow\mathsf{r}\\ \mathbf{Set}\end{array}$, and in $\mathbf{UFam}(A, A_\sharp)_1$ we of course have that if p and q are not empty (not the least element), then $p \wedge q$ is also not empty. This argument can be made precise as follows.

We first observe that the subobject classifier $\Omega = (PA, \supset\subset)$ in $\mathrm{RT}(A, A_\sharp)$ is a quotient of $\nabla_{\mathsf{r}}(PA)$:

$$\begin{array}{c}\nabla_{\mathsf{r}}(PA)\\ \Big\downarrow{\scriptstyle \supset\subset}\\ \Omega\end{array}$$

(this is just as for standard realizability toposes [Hyl82]). Further, the partial ordering $\leq_\Omega$ on Ω is induced by the subobject $R \rightarrowtail \nabla_{\mathsf{r}}(PA) \times \nabla_{\mathsf{r}}(PA)$ represented by the strict relation

$$\alpha, \beta\colon\ PA \mid R(\alpha, \beta) = \alpha \supset \beta.$$

Now, since Γ preserves both limits and colimits (as a right and left adjoint) we get that $\Lambda = \Gamma(\Omega)$ internally in $\mathrm{RT}(A, A_\sharp)$ is a quotient of $\Gamma(\nabla_{\mathsf{r}}(PA))$, which by Theorem 6.2.6 is isomorphic to $\nabla_{\mathsf{q}}(PA)$. In a diagram, we have

$$\begin{array}{c} \nabla_{\mathsf{q}}(PA) \\ \big\downarrow {\scriptstyle \asymp^\sharp} \\ \Lambda, \end{array}$$

where the quotient morphism is represented by $\alpha \asymp^\sharp \beta = (\alpha \asymp \beta) \cap A_\sharp$. Further we get that that the partial ordering on Λ is represented by the strict relation ΓR given by

$$\alpha, \beta\colon PA \mid \Gamma R(\alpha, \beta) = (\alpha \supset \beta) \cap A_\sharp,$$

where $\supset$ is the implication in the tripos underlying $\mathrm{RT}(A, A_\sharp)$. The internal conjunction map $\wedge_\Lambda$ on Λ is induced by the conjunction map

$$\wedge_{\nabla_{\mathsf{q}}(PA)}\colon \nabla_{\mathsf{q}}(PA) \times \nabla_{\mathsf{q}}(PA) \to \nabla_{\mathsf{q}}(PA)$$

represented by the functional relation

$$\alpha, \beta\colon PA \mid |\alpha \wedge_{\nabla_{\mathsf{q}}(PA)} \beta| \stackrel{\text{def}}{=} \alpha \wedge \beta,$$

where $\wedge$ is the conjunction in the tripos underlying $\mathrm{RT}(A, A_\sharp)$. To show that (10.3) is valid in $\mathrm{RT}(A_\sharp)$ it then suffices to show that

$$\forall \alpha, \beta\colon \nabla_{\mathsf{q}}(PA).\ \neg(\Gamma R(\alpha, \emptyset)) \wedge \neg(\Gamma R(\beta, \emptyset)) \supset \neg(\Gamma R(\alpha \wedge_{\nabla_{\mathsf{q}}(PA)} \beta, \emptyset))$$

is valid in $\mathrm{RT}(A_\sharp)$. Unfolding the definitions, we see that it suffices to show that

$$\forall \alpha, \beta\colon PA.\ (\neg(\Gamma R(\alpha, \emptyset)) \wedge (\neg(\Gamma R(\beta, \emptyset))) \supset (\neg(\Gamma R(\alpha \wedge_{\nabla_{\mathsf{q}}(PA)} \beta)))$$

is valid in the tripos q underlying $\mathrm{RT}(A_\sharp)$. But this is clearly the case, the identity $\lambda x.\, x$ is a realizer — note that the point is, as already mentioned above, that if α and β are both non-empty, then also $\alpha \wedge_{\nabla_{\mathsf{q}}(PA)} \beta$ is non-empty.

10.5 The Double-Negation Topology in $\mathrm{RT}(A, A_\sharp)$

In this section we present a number of definitions and facts about the $\neg\neg$-topology in $\mathrm{RT}(A, A_\sharp)$. The results are as one would expect, based on the experience with standard realizability toposes over PCA's. The proofs of the results are essentially the same as for a standard realizability topos over a PCA, and we therefore do not include them. See, *e.g.*, [HJP80, Pit81, Hyl82, RR90, Car95, Jac99] for the proofs for $\mathrm{RT}(A)$. In Subsection 10.5.1 we describe the interior operation explicitly on $\neg\neg$-separated objects. Furthermore, we give a concrete example of the interior operation which serves to show that we indeed can think of the interior operation as carving out the computable elements of an object, *cf.* Remark 10.1.1.

Convention 10.5.1. Unless otherwise mentioned, the topology referred to in this section is the $\neg\neg$-topology.

Recall that $\mathrm{RT}(A, A_\sharp)$ is obtained from the tripos $\begin{array}{c}\mathbf{UFam}(A,A_\sharp)\\ \downarrow r\\ \mathbf{Set}\end{array}$, see Chapter 6. Consider the geometric morphism $(\Gamma_{\mathsf{r}}, \nabla_{\mathsf{r}})\colon \mathbf{Set} \to \mathrm{RT}(A, A_\sharp)$ from Section 6.2. Recall that $\Gamma_{\mathsf{r}}(I, \approx) = \mathrm{Dom} \sim/\sim'$ where $\sim'$ is the least equivalence relation on $\mathrm{Dom} \sim$ containing $\sim$, with $i \sim i'$ iff $|i \approx i'| \neq \emptyset$. Further, recall that $\nabla_{\mathsf{r}}(X) = (X, \approx_X)$ where

$$|x \approx_X x'| = \begin{cases} A & \text{if } x = x', \\ \emptyset & \text{otherwise.} \end{cases}$$

and that $\nabla_{\mathsf{r}}(X) \cong (X, \approx'_X)$ where

$$|x \approx'_X x'| = \begin{cases} \{\mathbf{K}\} & \text{if } x = x', \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma 10.5.2. *The functor $\nabla\colon \mathbf{Set} \to \mathrm{RT}(A, A_\sharp)$ preserves the initial object.*

It follows, as for the Effective Topos [Hyl82], that **Set** is equivalent to the category of sheaves for the double negation topology $\neg\neg$ in $\mathrm{RT}(A, A_\sharp)$.

Let $(\varphi\colon I \to PA) \in \mathrm{SPred}(\mathrm{RT}(A, A_\sharp))$ be a strict predicate on $(I, \approx) \in \mathrm{RT}(A, A_\sharp)$. Then

$$\neg\neg\varphi = i \mapsto \begin{cases} E_I(i) & \text{if } \varphi(i) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus φ is **closed** iff

$$\exists a \in A_\sharp.\ \forall i \in I.\ \big(\varphi(i) \neq \emptyset \implies \forall b \in E_I(i).\ a(b) \in \varphi(i)\big).$$

Note that φ is **dense** if, for each $i \in I$,

$$E_I(i) \neq \emptyset \implies \varphi(i) \neq \emptyset.$$

Lemma 10.5.3. *The $\neg\neg$-closed subobjects of $(I, \approx) \in \mathrm{RT}(A, A_\sharp)$ correspond to subsets of $\Gamma_{\mathsf{r}}(I, \approx)$. More precisely, there is a change-of-base situation*

$$\begin{tikzcd} \mathrm{ClSub}_{\neg\neg}(\mathrm{RT}(A, A_\sharp)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sub}(\mathbf{Set}) \arrow[d] \\ \mathrm{RT}(A, A_\sharp) \arrow[r, "\Gamma_{\mathsf{r}}"'] & \mathbf{Set}. \end{tikzcd}$$

Definition 10.5.4. An object $(I, \approx_I) \in \mathrm{RT}(A, A_\sharp)$ is **canonically separated** if

$$|i \approx_I i'| \neq \emptyset \implies i = i' \qquad \text{and} \qquad E_I(i) \neq \emptyset$$

An object $(I, \approx_I) \in \mathrm{RT}(A, A_\sharp)$ is **canonically modest** if it is canonically separated and, moreover,

$$E_I(i) \cap E_I(i') \neq \emptyset \implies i = i'.$$

We often say that an object is canonically separated (or modest) if it isomorphic to a canonically separated (or modest) object.

Note that $\nabla_{\mathsf{r}}(X)$ is canonically separated, for all $X \in \mathbf{Set}$.

Lemma 10.5.5. *In the topos $\mathrm{RT}(A, A_\sharp)$ the following hold:*

1. *Every subobject of a canonically separated object is canonically separated.*
2. *Every subobject of a canonically modest object is canonically modest.*
3. *If $R \in \mathrm{SPred}((I, \approx) \times (I, \approx))$ is a closed equivalence relation on $(I, \approx)$, then (I, R) in the quotient $(I, \approx) \twoheadrightarrow (I, R)$ is canonically separated, and it is modest if $(I, \approx)$ is. Conversely, if (I, R) is canonically separated, then R is closed.*

Definition 10.5.6. Define $\mathbf{Asm}(A, A_\sharp)$ to be the subcategory of $\mathbf{Asm}(A)$ with objects all the objects of $\mathbf{Asm}(A)$ and with morphisms those morphism of $\mathbf{Asm}(A)$ that are tracked by a realizer from $A_\sharp$. That is, (X, E_X) is an object of $\mathbf{Asm}(A, A_\sharp)$ if X is a set and $E_X : X \to PA$ is a function such that $E_X(x) \neq \emptyset$, for all $x \in X$. Further, $f \colon (X, E_X) \to (Y, E_Y)$ is in $\mathbf{Asm}(A, A_\sharp)$ iff

$$x : X \mid E_X(x) \vdash E_Y(f(x))$$

is valid in tripos r.

The adjunction

$$\begin{tikzcd} \mathrm{RT}(A, A_\sharp) \arrow[r, shift left, "\Gamma_{\mathsf{r}}"] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, shift left, "\nabla_{\mathsf{r}}"] \end{tikzcd}$$

restricts to an adjunction

$$\begin{tikzcd} \mathbf{Asm}(A, A_\sharp) \arrow[r, shift left, "\Gamma_{\mathsf{r}}"] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, shift left, "\nabla_{\mathsf{r}}"] \end{tikzcd}$$

where $\nabla(X) = (X, E_X)$, with $E_X(x) = A$, and where $\Gamma_{\mathsf{r}}(X, E_X) = X$. We repeat that Γ_r is *not* isomorphic to the global sections functor

$$\mathrm{Hom}_{\mathbf{Asm}(A,A_\sharp)}(1, -).$$

Indeed $\mathrm{Hom}_{\mathbf{Asm}(A,A_\sharp)}(1, (X, E_X))$ is isomorphic to the underlying set of $\Gamma(X, E_X)$, *i.e.*, the set of elements x in X for which $E_X(x) \cap A_\sharp \neq \emptyset$.

Theorem 10.5.7. *The category* $\mathbf{Asm}(A, A_\sharp)$ *of assemblies is equivalent to the full subcategory of* $\mathrm{RT}(A, A_\sharp)$ *on the canonically separated objects. It is also equivalent to the full subcategory of separated objects.*

Definition 10.5.8. The category $\mathbf{Mod}(A, A_\sharp)$ of **modest sets over** A **with respect to** $A_\sharp$ is defined to be the full subcategory of $\mathbf{Asm}(A, A_\sharp)$ on the modest objects, *i.e.*, those (X, E_X) satisfying that $E_X(x) \cap E_X(x') \neq \emptyset \implies x = x'$, for all $x, x' \in X$.

Definition 10.5.9. The category $\mathbf{PER}(A, A_\sharp)$ of **partial equivalence relations over** A **with respect to** $A_\sharp$ is the category with

objects partial equivalence relations on A

morphisms $R \to S$ are equivalence classes of realizers $b \in A_\sharp$ satisfying

$$a \; R \; a' \implies b(a) \; S \; b(a')$$

with two such b and b' equivalent iff

$$a \; R \; a \implies b(a) \; S \; b'(a).$$

Proposition 10.5.10. *The categories* $\mathbf{PER}(A, A_\sharp)$ *and* $\mathbf{Mod}(A, A_\sharp)$ *are equivalent.*

The category $\mathbf{PER}(A, A_\sharp)$ is a small category and can be seen as an internal category in $\mathbf{Asm}(A, A_\sharp)$ whose externalization consists of families of partial equivalence relations with morphisms between such families tracked uniformly by computable realizers. The externalization is complete as a fibration (*i.e.*, has coproducts and fibred finite limits) and thus $\mathbf{PER}(A, A_\sharp)$ is a small internally complete category in $\mathbf{Asm}(A, A_\sharp)$. The results and their proofs are analogous to those for $\mathbf{PER}(A)$ and $\mathbf{Asm}(A)$, see, *e.g.*, [Jac99], and we do not include them here.

Definition 10.5.11. We define the following two **objects of realizers** in $\mathrm{RT}(A, A_\sharp)$:

$$A = (A, \approx_A) \qquad \text{and} \qquad A_\sharp = (A_\sharp, \approx_{A_\sharp})$$

where $|a \approx_A a'| = \{a\} \cap \{a'\}$ and $|a \approx_{A_\sharp} a'| = \{a\} \cap \{a'\}$.

Note that these two objects are both modest. We often simply write them as (A, E_A) with $E_A(a) = \{a\}$ and $(A_\sharp, E_{A_\sharp})$ with $E_{A_\sharp}(a) = \{a\}$.

Proposition 10.5.12. *The category* $\mathbf{PER}(A, A_\sharp)$ *is equivalent to the full subcategory of* $\mathrm{RT}(A, A_\sharp)$ *on the separated subquotients* (X, E_X) *of (closed) subobjects* $(Y, \approx)$ *of the object of realizers* A*.*

$$\begin{tikzcd} (Y, \approx) \arrow[r, tail] \arrow[d] & (A, E_A) \\ (X, E_X) & \end{tikzcd}$$

Lemma 10.5.13. $\nabla\colon \mathbf{Set} \to \mathrm{RT}(A, A_\sharp)$ *preserves epimorphism.*

Proposition 10.5.14. *The category of* $\mathbf{Mod}(A, A_\sharp)$ *is a reflective subcategory of* $\mathbf{Asm}(A, A_\sharp)$*, which again is a reflective subcategory of* $\mathrm{RT}(A, A_\sharp)$*. Both of the reflectors preserve products, so both inclusions form exponential ideals.*

10.5.1 Interior and $\neg\neg$-separated Objects

For $\neg\neg$-separated objects the interior is nothing but the composite of functors Δ and Γ:

Proposition 10.5.15. *Let* $(X, \approx_X) \in \mathrm{RT}(A, A_\sharp)$ *be* $\neg\neg$*-separated. Then the interior* $(X, \approx_X)^\circ$ *of* $(X, \approx_X)$ *is isomorphic to* $\Delta\Gamma(X, \approx_X)$*.*

Proof. Recall from Section 10.1.4 that the interior of $(X, \approx_X)$ is obtained by taking the image of the counit $\Delta\Gamma(X, \approx_X) \to (X, \approx_X)$ of $\Delta \dashv \Gamma$. Therefore it clearly suffices to show that this counit is monic. Recall that the counit is represented by the functional relation

$$E(x, x') = (x \approx_X x) \cap A_\sharp \wedge (x \approx_X x').$$

Using that we (by Theorem 10.5.7) may assume that $(X, \approx_X)$ is canonically separated, it is easy to see that E represents a monomorphism, *i.e.*,

$$x_1, x_2, x : X \mid E(x_1, x) \wedge E(x_2, x) \vdash (x_1 \approx_X x_2) \cap A_\sharp$$

is valid in the logic of the tripos r underlying $\mathrm{RT}(A, A_\sharp)$ (it is realized by the $A_\sharp$-realizer $\lambda x.\ \pi(\pi(x))$.) □

Using the above proposition we now embark on developing a concrete example (see Example 10.5.19 below), which shows that the interior of an object X in $\mathrm{RT}(A, A_\sharp)$ really does consist of the computable elements of X, as hinted at in Remark 10.1.1. To this end we first establish a couple of simple facts concerning exponentials.

Recall the following standard proposition concerning exponentials in cartesian closed categories related by an adjunction.

Proposition 10.5.16. *Let $\mathbb{C}$ and $\mathbb{D}$ be cartesian categories, with $\mathbb{D}$ closed, and let*

$$\mathbb{C} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{D}$$

be an adjunction with F full and faithful and product-preserving. Then X^Y in $\mathbb{C}$ is isomorphic to $G(FX^{FY})$.

Proof.

$$\begin{aligned} \mathbb{C}(Z, G(FY^{FX})) &\cong \mathbb{D}(FZ, FY^{FX}) \\ &\cong \mathbb{D}(FZ \times FX, FY) \\ &\cong \mathbb{D}(F(Z \times X), FY) \\ &\cong \mathbb{C}(Z \times X, Y). \end{aligned}$$

□

Corollary 10.5.17. *For objects $X, Y \in \mathrm{RT}(A_\sharp)$, the exponential Y^X in $\mathrm{RT}(A_\sharp)$ is isomorphic to $\Gamma((\Delta Y)^{(\Delta X)})$.*

Corollary 10.5.18. *For $\neg\neg$-separated objects $X, Y \in \mathrm{RT}(A_\sharp)$, the exponential X^Y in $\mathrm{RT}(A_\sharp)$ is isomorphic to the interior of the exponential of X and Y in $\mathrm{RT}(A, A_\sharp)$.*

Proof. Formally the proposition says that $\Delta(X^Y) \cong ((\Delta X)^{(\Delta Y)})^\circ$ in the topos $\mathrm{RT}(A, A_\sharp)$. This follows by Corollary 10.5.17 and Proposition 10.5.15. □

Example 10.5.19. The point of this example is to demonstrate that, in a concrete case, the interior of an object indeed consists of the computable elements, *cf.* Remark 10.1.1.

Let A be the graph model PCA $\mathbb{P}$ from Example 3.1.15 and let $A_\sharp$ be the r.e. sub-model RE of $\mathbb{P}$ from Example 3.1.16.

The natural numbers object N in $\mathrm{RT}(RE)$ is the modest set (N, E_N) with $E_N(n) = \{\{n\}\}$. This is then also the natural numbers object in $\mathrm{RT}(\mathbb{P}, RE)$ and in $\mathrm{RT}(\mathbb{P})$.

The exponential N^N of the natural numbers object N in $\mathrm{RT}(RE)$ is the modest set (X, E_X) with underlying set X the set of recursive functions from the natural numbers to the natural numbers, and with each $E_X(f)$ the set of elements of RE tracking f.

The exponential N^N in $\mathrm{RT}(\mathbb{P}, RE)$ is the same as in $\mathrm{RT}(\mathbb{P})$, namely (N^N, E) with N^N the exponential in **Set**, *i.e.*, the set of *all* set-theoretic functions from the natural numbers to the natural numbers, and with $E(f)$ the set of elements of $\mathbb{P}$ tracking f.

By Corollary 10.5.18 we see that N^N in $\mathrm{RT}(RE)$ is isomorphic to the interior of N^N in $\mathrm{RT}(\mathbb{P})$. Thus we see that the interior indeed carves out the computable elements of N^N.

10.6 $\mathrm{RT}(A, A_\sharp)$ as an Exact Completion

In this section we show that $\mathrm{RT}(A, A_\sharp)$ can be described as the exact completion of its full subcategory of projectives. The development is analogous to the one for $\mathrm{RT}(A)$ in [RR90] and the proofs of the results are essentially the same as for $\mathrm{RT}(A)$ and we therefore leave out most of them. We assume familiarity with the exact and the regular completion of a left exact category [Car95, CV98]).

Definition 10.6.1. We define **PartAsm**$(A, A_\sharp)$, the category of **partitioned assemblies over A with respect to $A_\sharp$**, to be the category with objects surjective functions $\sigma\colon X \twoheadrightarrow I$ in **Set**, where $I \subseteq A$, and with morphisms from $\sigma\colon X \twoheadrightarrow I$ to $\tau\colon Y \twoheadrightarrow J$ functions from X to Y in **Set** for which there exists an $A_\sharp$-definable function g such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "\sigma"'] & Y \arrow[d, two heads, "\tau"] \\ I \arrow[r, "g"'] & J \end{tikzcd}$$

commutes in **Set**.

Note that $\mathbf{PartAsm}(A, A_\sharp)$ is a full subcategory of $\mathbf{Asm}(A, A_\sharp)$, so the image of a partitioned assembly under the inclusion functor

$$\mathbf{PartAsm}(A, A_\sharp) \hookrightarrow \mathbf{Asm}(A, A_\sharp) \hookrightarrow \mathrm{RT}(A, A_\sharp)$$

is a separated object.

Proposition 10.6.2. *An object X of $\mathrm{RT}(A, A_\sharp)$ is projective iff it is isomorphic to one of the form $(P, \approx_P)$ where, for any $p, p' \in P$, $|p \approx_P p'|$ contains at most one element, and where if $|p \approx_P p| = \{a\}$, then also $|p \approx_P p'| = \{a\}$ for any p' such that $|p \approx_P p'|$ is non-empty.*

The proof of the proposition proceeds in the appendix of [RR90, Appendix] and uses the following lemma.

Lemma 10.6.3. *Any separated object $(X, \approx_X)$ in $\mathrm{RT}(A, A_\sharp)$ is covered by a projective $(Q, \approx_Q)$ which is a subobject of $(X, \approx_X) \times (A, \approx_A)$, as in the diagram*

$$\begin{tikzcd} (Q, \approx_Q) \arrow[r, tail] \arrow[d, two heads] & (X, \approx_X) \times (A, \approx_A) \\ (X, \approx_X). & \end{tikzcd}$$

Proof. By Theorem 10.5.7, we may assume that $(X, \approx_X)$ is canonically separated. Then define object $(Q, \approx_Q)$ where $Q = \{\, (x, a) \mid x \in X \text{ and } a \in |x \approx_X x| \,\}$ and

$$|(x, a) \approx_Q (x', a')| = \begin{cases} \{a\} & \text{if } x = x' \text{ and } a = a', \\ \emptyset & \text{otherwise.} \end{cases}$$

Then we have the required diagram. $\square$

Corollary 10.6.4. $\mathrm{RT}(A, A_\sharp)$ *has enough projectives: Let* X *be an arbitrary object of* $\mathrm{RT}(A, A_\sharp)$. *Then there exists a projective* $P \in \mathrm{RT}(A, A_\sharp)$, *which covers* X. *(*i.e., *there is a cover* $P \twoheadrightarrow X$*).*

Proof. By the previous lemma, using that any object in $\mathrm{RT}(A, A_\sharp)$ is covered by a separated object. □

Corollary 10.6.5. *The full subcategory of* $\mathrm{RT}(A, A_\sharp)$ *on the projective objects is closed under finite limits.*

Proposition 10.6.6. *The full subcategory of* $\mathrm{RT}(A, A_\sharp)$ *on the projective objects is equivalent to the category* $\mathbf{PartAsm}(A)$ *of partitioned assemblies.*

Theorem 10.6.7.

1. *The category* $\mathbf{Asm}(A, A_\sharp)$ *is equivalent to* $(\mathbf{PartAsm}(A, A_\sharp))_{reg/lex}$.
2. *The category* $\mathrm{RT}(A, A_\sharp)$ *is equivalent to* $(\mathbf{PartAsm}(A, A_\sharp))_{ex/lex}$ *and to* $(\mathbf{Asm}(A, A_\sharp))_{ex/reg}$.

Corollary 10.6.8. *The following objects in* $\mathrm{RT}(A, A_\sharp)$ *are projective:*

1. *The objects of realizers* (A, E_A) *and* $(A_\sharp, E_{A_\sharp})$.
2. *The terminal object* 1.
3. *Any sheaf,* i.e., *any object in the image of* $\nabla \colon \mathbf{Set} \to \mathrm{RT}(A, A_\sharp)$.

Since all objects in **Set** are projective, we can rephrase item 3 in the above corollary as "the functor $\nabla \colon \mathbf{Set} \to \mathrm{RT}(A, A_\sharp)$ preserves projectives."

Proposition 10.6.9. *An object* X *in* $\mathrm{RT}(A, A_\sharp)$ *is projective iff it is internally projective.*

Proof. Since 1 is projective, if X is internally projective then X is projective. The other direction follows from the proof of the characterization of the projectives, see [RR90, Appendix]. □

We write $|X|$ for the cardinality of a set X. The **cardinality of a partitioned assembly** (X, E_X) is defined to be the cardinality of X.

The following is an easy generalization of a recent observation of Jaap van Oosten, who showed that the countable partitioned assemblies generate the effective topos.

Proposition 10.6.10. *The partitioned assemblies of cardinality less than or equal to* $|A_\sharp|$ *generate* $\mathrm{RT}(A, A_\sharp)$.

Proof. Since every object is covered by a projective and the partitioned assemblies are the projectives, it suffices to show that, for any pair of morphisms $f, g \colon X \to Y$ in $\mathrm{RT}(A, A_\sharp)$ such that $f \neq g$ and where X is a partitioned assembly $(|X|, E_X)$, there exists a partitioned assembly P with cardinality less than $|A_\sharp|$ and a map $i \colon P \to X$ such that $f \circ i \neq g \circ i$. Suppose given such X, Y, f, and g and write F and G for functional relations representing f and g, respectively. Then $f \neq g$ means that

$$A_\sharp \cap \left(\bigcap_{x \in |X|, y \in |Y|} (F(x,y) \supset G(x,y)) \right) \neq \emptyset.$$

Thus for each $a \in A_\sharp$, there exists a $x_a \in |X|$ such that $a \notin \bigcap_{y \in |Y|} (F(x,y) \supset G(x,y))$. Choose such an x_a, for each $a \in A_\sharp$, and let $|P|$ be the set of all such chosen x_a's. Let $E_P(x) = E_X(x)$. Then $P = (|P|, E_P)$ is a partitioned assembly with cardinality less than or equal to $|A_\sharp|$ and

$$A_\sharp \cap \left(\bigcap_{x \in |P|, y \in |Y|} (F(x,y) \supset G(x,y)) \right) \neq \emptyset.$$

Thus letting $i \colon P \to X$ be the obvious inclusion map, we have that $f \circ i \neq g \circ i$, as required. □

10.7 Logical Principles

In this section we consider the question of which of the logical principles discussed in constructive mathematics [TvD88] are valid.

Recall that in any PCA, one can represent the natural numbers, see, *e.g.*, [Lon94, Section 1.1.3, Page 35] for a convenient representation in terms of Curry numerals. Using this representation one finds that any realizability topos has a natural numbers object N, which is a modest set with underlying set the ordinary set of natural numbers and with each natural number n realized by the set $\{\overline{n}\}$, where $\overline{n}$ is the representation of n in the PCA. See, *e.g.*, [Lon94, Chapter 1] for more details. Recall that the inclusion $\Delta \colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ preserves natural numbers object, as the inverse image of a geometric morphism. Therefore, the natural numbers object N in $\mathrm{RT}(A_\sharp)$, represented as just described, is also a natural numbers object in $\mathrm{RT}(A, A_\sharp)$.

Proposition 10.7.1. *The internal axiom of choice*

$$AC(N,X) \qquad \forall n \colon N.\ \exists x \colon X.\ \varphi(n,x) \supset \exists f \colon X^N.\ \forall n \colon N.\ \varphi(n, f(n))$$

(for $\varphi \rightarrowtail N \times X$ any subobject of $N \times X$) is valid in $\mathrm{RT}(A, A_\sharp)$.

Proof. Observe that N is a partitioned assembly. Therefore, by Proposition 10.6.6, N is projective and thus, by Proposition 10.6.9, also internally projective, and hence $\mathrm{AC}(N, X)$ is valid. The proof can carried out explicitly in the same manner as in [Hyl82, Section 9]. □

Proposition 10.7.2. *Markov's principle*

$$\mathrm{MP} \qquad \forall R\colon PN.\ \big((\forall n.\ R(n) \vee \neg R(n)) \wedge \neg\neg(\exists n.\ R(n))\big) \supset \exists n.\ R(n)$$

is valid in the internal logic of $\mathrm{RT}(A, A_\sharp)$.

Proof. The proof is as in Phoa [Pho93, Page 90] (Phoa gives the proof for a schema, with no quantification over R, but the same proof works in our case). □

Proposition 10.7.3. *The uniformity principle*

$$\mathrm{UP} \qquad \forall\varphi\big(\forall X\colon PN.\ \exists n\colon N.\ \varphi(X, n) \supset \exists n\colon N.\ \forall X\colon PN.\ \varphi(X, n)\big)$$

is valid in the internal logic of $\mathrm{RT}(A, A_\sharp)$.

Proof. The proof is as in [Hyl82, Section 15] (after noting that also Proposition 14.3 in [Hyl82] carries over to $\mathrm{RT}(A, A_\sharp)$). □

Proposition 10.7.4. *The arithmetical form of Church's thesis,* CT_0 *(see Definition 8.3.7) is not always valid in the internal logic of* $\mathrm{RT}(A, A_\sharp)$, i.e., *there exist PCA's A and $A_\sharp$ such that* $\mathrm{RT}(A, A_\sharp)$ *does not validate* CT_0.

Proof. Let $A = \mathbb{P}$ and $A_\sharp = RE$, see Examples 3.1.15 and 3.1.16. Suppose $\mathrm{RT}(A, A_\sharp) = \mathrm{RT}(\mathbb{P}, RE)$ satisfies CT_0. Then since the logical functor $Q\colon \mathrm{RT}(\mathbb{P}, RE) \to \mathrm{RT}(\mathbb{P})$ preserves first-order arithmetic also $\mathrm{RT}(\mathbb{P})$ validates CT_0, a contradiction, see the proof of Theorem 10.3.1. □

Chapter 11

Conclusion and Further Research

We have suggested a general notion of realizability based on weakly closed partial cartesian categories. We have shown that any weakly closed partial cartesian category gives rise to categories of assemblies and modest sets which model dependent predicate logic. The framework includes both standard realizability over partial combinatory algebras and also realizability over typed models, such as the category of algebraic lattices. In particular, the category **Equ** of equilogical spaces arises as modest sets over the category of algebraic lattices and, therefore, **Equ** models dependent predicate logic. As an application of the theory, we have detailed the interpretation of dependent predicate logic in **Equ** in concrete terms. Further, we have characterized when a weakly closed partial cartesian category gives rise to a topos; it happens just in case the weakly closed partial cartesian category has a universal object of which all other objects are retracts.

We have initiated a study of the relative realizability topos $\mathrm{RT}(A, A_\sharp)$ and shown that there is a localic local map of toposes from it to the standard realizability topos $\mathrm{RT}(A, A_\sharp)$ over $A_\sharp$, for $A_\sharp$ a sub partial combinatory algebra of A. We have provided a complete axiomatization of local maps of toposes and studied the connection between internal logics of toposes connected via a local map. Moreover, we have developed a modal logic for local maps. Special emphasis has been given to localic local maps, which have been described both in terms of internal locales and in terms of local triposes. We have shown how the modal logic is interpreted in the relative realizability model $\mathrm{RT}(A, A_\sharp)$. An alternative view of $\mathrm{RT}(A, A_\sharp)$ has been provided by showing that it arises as an exact completion in much the same way as standard realizability toposes does. The double-negation topology in $\mathrm{RT}(A, A_\sharp)$ has been studied and the validity of a couple of logical principles has been established. We have observed that $\mathrm{RT}(A)$ arises as a filter-quotient of $\mathrm{RT}(A, A_\sharp)$ and that the filter is given internally on that internal locale in $\mathrm{RT}(A_\sharp)$ for which $\mathrm{RT}(A, A_\sharp)$ is the category of $\mathrm{RT}(A_\sharp)$-valued sheaves.

There are many avenues for further research. We have already mentioned some of them regarding the general notion of realizability, see Chapters 3 and 4. We now indicate some other directions for further research regarding relative realizability and local maps.

11.1 Axioms for Local Maps of Toposes

It would be interesting to investigate if our axioms for local maps of toposes could be generalized to local maps of pretoposes. This seems plausible since our proofs mostly use elementary exactness properties and elementary properties of sheaves which should also hold for closure operators on pretoposes. Naturally, one would then also like to know how the results concerning logic and local maps generalizes to, say, Heyting pretoposes. Models could probably be found by considering realizability over WCPC-categories with morphisms definable via simply-typed lambda calculus.

As already hinted at, we hope that one can also use our axioms for local maps as a first step towards a suitable axiomatic definition of gros toposes [Law86]. As a first concrete question one may ask under what extra axioms (besides those for local maps) does the inclusion of discrete objects have a left exact left adjoint?

11.2 Logic and Local Maps of Toposes

We have not yet begun to investigate questions of completeness for the modal logic for local maps. Suppose given a theory in the modal logic. A good question then is whether we then construct a local map of toposes such that the resulting model of the modal logic validates exactly the provable sentences of the given theory ? Of course, the same question may be asked for localic local maps. Other questions include:

- When does the tripos-to-topos construction applied to $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$ yield $\mathcal{E}$? We know that it does when $\mathcal{E}$ is localic over $\mathrm{D}_j\mathcal{E}$, but it may happen more generally.

- What can be said in general about the logic for a local map when the subobject classifier is open?

11.3 Relative Realizability

Regarding relative realizability, the most pressing next step is to explore some concrete models and make use of the general results established here. As mentioned before, Andrej Bauer is currently investigating the model based on the graph model and its recursive enumerable submodel, see his forthcoming thesis [Bau00].

Another interesting point of view which we have not discussed before in this thesis is to consider $A_1 = (A_\sharp \rightarrowtail A)$ as an *internal* partial combinatory algebra in $\mathbf{Set}^{\rightarrow}$. This was suggested to me by Jaap van Oosten. One can show, as van Oosten pointed out to me, that $\mathrm{RT}(A, A_\sharp)$ is equivalent to the topos obtained by the tripos-to-topos construction applied to the $\mathbf{Set}^{\rightarrow}$-tripos that results from taking the $\neg\neg$-closed subsets of the internal PCA A_1. The toposes $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A)$ arise in the same way from the internal PCA's $A_0 = (A_\sharp \to A_\sharp)$ and $A_2 = (A \to A)$ in $\mathbf{Set}^{\rightarrow}$. This viewpoint allows for an alternative proof of the existence of the logical functor from $\mathrm{RT}(A, A_\sharp)$ to $\mathrm{RT}(A)$. Pursuing this point of view a bit further, I have found that $\mathrm{RT}(A, A_\sharp)$ is an open subtopos of the realizability topos constructed over the internal PCA A_1. (A similar result was proved by van Oosten, who showed that the effective topos is an open subtopos of the realizability topos constructed over the internal PCA $(N \to N)$ in $\mathbf{Set}^{\rightarrow}$, see [vO97b].) By taking $\mathrm{RT}(A, A_\sharp)$'s closed complement one gets a new topos, which may reasonably be called the *modified relative realizability topos* since it arises in the same way as the (standard) modified realizability topos arises [vO97b]. These and other related results will be described in detail elsewhere.

Appendix A

Dependent Type Theory and Predicate Logic in Equ

In this appendix I present the calculus of dependent type theory and predicate logic and sketch how it is modelled in $\mathbf{Equ} \simeq \mathbf{Mod}(\mathbf{ALat})$. In Chapters 3 and 4 we have already *proved* that **Equ** models dependent predicate logic with full dependent subset types and with quotient types (see Theorems 3.6.20, 4.1.3, 4.4.1, 4.2.1, and 4.3.1 which apply to $\mathbf{Asm}(\mathbf{ALat})$ and Section 3.7 which, together with the remarks on Page 90, says that the relevant results for $\mathbf{Asm}(\mathbf{ALat})$ also hold for $\mathbf{Mod}(\mathbf{ALat})$.) Here we merely work out what the interpretation is in concrete terms. It is basically straightforward to do so using the above mentioned theorems and the explanation in [Jac99] for how to interpret dependent type theory and predicate logic in DPL-structures. We have nevertheless chosen to include this treatment for the following reasons. First, we hope it may make the abstract treatment in Chapters 3 and 4 more accessible to readers not familiar with [Jac99]. Indeed it may be helpful to read this appendix in parallel with the treatment in Chapters 3 and 4. Second, we note that when one wants to *use* the type theory and logic to construct objects or prove properties about the model, one often needs to know what the interpretation is in concrete terms and so it makes sense to actually work it out.

We describe the syntax of the dependent type theory and predicate logic and sketch the interpretation of it in the DPL-structure

$$\begin{tikzcd}
\mathbf{UFam}(\mathbf{ALat}) \arrow[dr] & \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat})) \arrow[r, "\mathcal{P}"] \arrow[d] & \mathbf{Mod}(\mathbf{ALat})^{\rightarrow} \arrow[dl, "\mathrm{cod}"] \\
& \mathbf{Mod}(\mathbf{ALat}). &
\end{tikzcd}$$

In fact, to understand this appendix, it is not necessary to know what a DPL-structure is; it suffices to know the definitions of the categories $\mathbf{Mod}(\mathbf{ALat})$, $\mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))$ and $\mathbf{UFam}(\mathbf{ALat})$ and also the definition of the action of the functor $\{-\}\colon \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat})) \to \mathbf{Mod}(\mathbf{ALat})$ on objects (Definitions 3.7.3, 3.6.14 and 4.4.2). To make this appendix self-contained, however, we repeat those definitions here.

For a set A we write PA for the powerset of A. For A an algebraic lattice, we do not distinguish notationally between A and its underlying set. Category $\mathbf{Mod}(\mathbf{ALat})$ has as objects triples of the form (X, A, E) with $X \in \mathbf{Set}$, $A \in \mathbf{ALat}$, and $E\colon X \to PA$ a function in $\mathbf{Set}$ satisfying that whenever $x \neq x'$, we have that $E(x) \cap E(x') = \emptyset$. Morphisms $f\colon (X, A, E) \to (Y, B, E')$ are functions $f\colon X \to Y$ in $\mathbf{Set}$ for which there exists a $g \in B^A$ (*i.e.*, a continuous function from A to B in $\mathbf{ALat}$) such that $\forall x \in X.\ \forall a \in E(x).\ g(a) \in E'(f(x))$. Composition and identities are as in $\mathbf{Set}$. For such functions $E\colon X \to PA$ and $E'\colon Y \to PB$ we often write $E(x) \wedge E'(y)$ for $\{\, \langle a, b\rangle \mid a \in E(x) \text{ and } b \in E'(y) \,\}$.

Category $\mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))$ has as objects triples of the form

$$(I, A, (X_i, E_i)_{i \in X_I})$$

with $I = (X_I, A_I, E_I) \in \mathbf{Mod}(\mathbf{ALat})$ and $(X_i, A, E_i) \in \mathbf{Mod}(\mathbf{ALat})$, for all $i \in X_I$. A morphism $(I, A, (X_i, E_i)_{i \in X_I}) \to (J, B, (Y_i, E'_i)_{i \in X_J})$, with $J = (X_J, A_J, E_J)$, is a pair $(f, (f_i)_{i \in X_I})$ such that $f\colon I \to J$ in $\mathbf{Mod}(\mathbf{ALat})$ and such that there exists a $g\colon B^{A_I \times A}$ satisfying

$$\forall i \in X_I.\ \forall a_i \in E_I(i).\ \forall x \in X_i.\ \forall a \in E_i(x).\ g(a_i, a) \in E'_{f(i)}(f_i(x)).$$

The identity is $(id, (id)_{i \in X_I})$ and the composition of

$$(f, (f_i)_{i \in X_I}) \qquad \text{and} \qquad (g, (g_j)_{j \in X_J})$$

is $(gf, (g_{f(i)} f_i)_{i \in X_I})$.

Category $\mathbf{UFam}(\mathbf{ALat})$ has as objects pairs of the form $(I, (B, \varphi))$ with $I = (X_I, A_I, E_I) \in \mathbf{Mod}(\mathbf{ALat})$ and (B, φ) an equivalence class of "predicates," where $B \in \mathbf{ALat}$, and $\varphi\colon X_I \to PB$ in $\mathbf{Set}$, with two such (B, φ)

and (C, ψ) equivalent iff there exist continuous functions $g \in C^{A \times B}$ and $h \in B^{A \times C}$ such that

$$\forall i \in X_I.\ \forall a \in E_I(i).\ \forall b \in \varphi(i).\ g(a, b) \in \psi(i)$$

and

$$\forall i \in X_I.\ \forall a \in E_I(i).\ \forall c \in \psi(i).\ h(a, c) \in \varphi(i).$$

A morphism $u\colon (I, (B, \varphi)) \to (J, (C, \psi))$ in $\mathbf{UFam}(\mathbf{ALat})$ is a morphism $u\colon I \to J$ in $\mathbf{Mod}(\mathbf{ALat})$ for which there exists a continuous function $g\colon C^{A_I \times B}$ such that

$$\forall i \in X_I.\ \forall a \in E_I(i).\ \forall b \in \varphi(i).\ g(a, b) \in \psi(u(i)).$$

Finally, functor $\{-\}\colon \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat})) \to \mathbf{Mod}(\mathbf{ALat})$ is defined as follows. On an object $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$ with $I = (X_I, A_I, E_I)$, $\{\mathcal{X}\} = \big(\coprod_{i \in X_I} X_i, A_I \times A, E\big)$ with $E(i, x) = E_I(i) \wedge E_i(x) = \{\, \langle a, b \rangle \mid a \in E_I(i) \text{ and } b \in E_i(x) \,\}$. On a morphism $(f, (f_i)_{i \in X_I})$, $\{(f, (f_i)_{i \in X_I})\} = (i, x) \mapsto (f(i), f_i(x))$.

Sequents and Their Interpretation Sequents have one of the following forms:

1. $\Gamma\colon \mathsf{Ctx}$
2. $\Gamma \vdash \sigma\colon \mathsf{Type}$
3. $\Gamma \vdash M\colon \sigma$
4. $\Gamma \vdash M = N\colon \sigma$
5. $\Gamma \vdash \sigma = \tau\colon \mathsf{Type}$
6. $\Gamma \vdash \varphi\colon \mathsf{Prop}$
7. $\Gamma \mid \Theta \vdash \varphi$

Sequent 1 is used for context formation. A context Γ is interpreted as an object I in the base category $\mathbf{Mod}(\mathbf{ALat})$ of the DPL-structure. For the remainder of this paragraph suppose that Γ is interpreted by $I = (X_I, A_I, E_I) \in \mathbf{Mod}(\mathbf{ALat})$. Sequent 2 is for type formation. A type σ in context Γ, written $\Gamma \vdash \sigma\colon \mathsf{Type}$, is interpreted as an object $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$ in $\mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_I$, *i.e.*, in the fibre over I. For the remainder of this paragraph suppose that $\Gamma \vdash \sigma\colon \mathsf{Type}$ is interpreted by this $\mathcal{X} \in \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_I$. Sequent 3 is for term formation. A term $\Gamma \vdash M\colon \sigma$ is interpreted as a morphism $1(I) \to \mathcal{X}$ in the fibre $\mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_I$ over I. Equivalently [Jac99, Page 619], the term is interpreted as a section to the projection $\pi_{\mathcal{X}}\colon \{\mathcal{X}\} \to I$ in $\mathbf{Mod}(\mathbf{ALat})$. Sequent 4 is conversion equality of terms; if $\Gamma \vdash M = N\colon \sigma$, then the terms $\Gamma \vdash M\colon \sigma$ and $\Gamma \vdash N\colon \sigma$ are interpreted as equal morphisms. Likewise, sequent 5 is for type equality; if $\Gamma \vdash \sigma = \tau\colon \mathsf{Type}$, then the types $\Gamma \vdash \sigma\colon \mathsf{Type}$ and $\Gamma \vdash \tau\colon \mathsf{Type}$ are interpreted by the same objects in the fibre $\mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_I$ over I. Sequent 6 is for formation of propositions in context. A proposition in context $\Gamma \vdash \varphi\colon \mathsf{Prop}$ is interpreted as an object φ in the fibre $\mathbf{UFam}(\mathbf{ALat}))_I$ over I, *i.e.*, as an equivalence class (B, φ) with $\varphi\colon X_I \to PB$. Note that we do not distinguish notationally between syntax and semantics here. We often omit the underlying algebraic lattice of realizers B from (B, φ) when it is clear from context. For the remainder of this paragraph, suppose that the proposition in context $\Gamma \vdash \varphi\colon \mathsf{Prop}$ is interpreted by object $(B, \varphi) \in \mathbf{UFam}(\mathbf{ALat})_I$. Finally, the sequent 7 is for logical entailment. In $\Gamma \mid \Theta \vdash \varphi$, the Θ is a sequence $(\varphi_1, \ldots, \varphi_m)$ of propositions in context Γ with $m \geq 0$ (if $m = 0$, we write $\Theta = \emptyset$ and the interpretation is $\Theta = \top$ — see below for the interpretation of $\top$), and sequent $\Gamma \mid \Theta \vdash \varphi$ is interpreted as $(\varphi_1 \wedge \cdots \wedge \varphi_m) \leq_I \varphi$ in the fibre over I in $\mathbf{UFam}(\mathbf{ALat})$. Thus $\Gamma \mid \Theta \vdash \varphi$ is **valid** or **holds** iff $(\varphi_1 \wedge \cdots \wedge \varphi_m) \leq_I \varphi$ in $\mathbf{UFam}(\mathbf{ALat})_I$.

Convention A.0.1. For an object $I = (X_I, A_I, E_I) \in \mathbf{Mod}(\mathbf{ALat})$ we sometimes write $|I|$ for the underlying set X_I of I. Moreover, when given objects $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$ and $\mathcal{Y} = (I, B, (Y_i, E'_i)_{i \in X_I})$ in the category $\mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_I$, we will often omit the prime on E'_i and just write $\mathcal{Y}$ as $(I, B, (Y_i, E_i)_{i \in X_I})$; it will be clear from context which E_i is meant.

We present the rules and their interpretation in the following order. First we cover the rules of basic dependent type theory, then the dependent predicate logic and finally dependent subset and quotient types. But first a couple of remarks concerning the conversion sequents for terms, 4, and types, 5: We leave out the rules making term conversion and type conversion into congruence relations. Then the only remaining rule associated with type conversion is the following:

$$\frac{\Gamma \vdash M\colon \sigma \qquad \Gamma \vdash \sigma = \tau\colon \mathsf{Type}}{\Gamma \vdash M\colon \tau}\ \textbf{conversion}$$

Its interpretation is trivial (the interpretation of M stays the same) since $\Gamma \vdash \sigma \colon \mathsf{Type}$ and $\Gamma \vdash \tau \colon \mathsf{Type}$ are interpreted by the same semantic objects. This completes the description of the interpretation of the sequents for term conversion and type conversion; below we include the basic rules for term conversion.

Rules for Context Formation The start rule for context formation is

$$\frac{}{\emptyset \colon \mathsf{Ctx}}$$

which is interpreted by the terminal object $1 \in \mathbf{Mod}(\mathbf{ALat})$. There is one other rule for context formation:

$$\frac{\Gamma \vdash \sigma \colon \mathsf{Type}}{(\Gamma, x \colon \sigma) \colon \mathsf{Ctx}}$$

with the side-condition that x is assumed not to be bound in Γ (this side-condition is completely standard and we shall not be more precise about it and leave it implicit in the following). Suppose Γ is interpreted by $I = (X_I, A_I, E_I) \in \mathbf{Mod}(\mathbf{ALat})$ and that $\Gamma \vdash \sigma \colon \mathsf{Type}$ is interpreted by $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I}) \in \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_I$. Then $(\Gamma, x \colon \sigma) \colon \mathsf{Ctx}$ is interpreted as $\{\mathcal{X}\} \in \mathbf{Mod}(\mathbf{ALat})$. We recall that $\{\mathcal{X}\}$ equals $(\coprod_{i \in X_I} X_i, A_I \times A, E)$ with $E(i,x) = E_I(i) \wedge E_i(i) = \{ \langle a, b \rangle \mid a \in E_I(i) \text{ and } b \in E_i(x) \}$.

Structural Rules for Dependent Type Theory We now consider the basic structural rules of dependent type theory. The interpretation is straightforward, although notationally a bit messy.

The projection rule

$$\frac{\Gamma \vdash \sigma \colon \mathsf{Type}}{\Gamma, x \colon \sigma \vdash x \colon \sigma} \,\textbf{projection}$$

is interpreted as follows. Suppose Γ is interpreted as $I = (X_I, A_I, E_I)$ and $\Gamma \vdash \sigma \colon \mathsf{Type}$ is interpreted as $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$. Then the term $\Gamma, x \colon \sigma \vdash x \colon \sigma$ is interpreted as the morphism

$$\big((i,x) \mapsto i, (* \mapsto x)_{(i,x) \in \coprod_{i \in X_I} X_i}\big)$$

from the terminal object over $\{\mathcal{X}\}$ to $\mathcal{Y}$ in the fibre over $\{\mathcal{X}\}$, where $\mathcal{Y} = (\{\mathcal{X}\}, A, (X_i, E_i)_{(i,x) \in \coprod_{i \in X_I} X_i})$. Equivalently, the term is interpreted as the section $(i,x) \mapsto ((i,x), x)$ from $\{\mathcal{X}\}$ to $\{\mathcal{Y}\}$ in $\mathbf{Mod}(\mathbf{ALat})$.

In the following, let $\mathcal{J}$ stand for an arbitrary expression which may occur on the right of a turnstile Γ in one of the sequents 2, 3, 4, or 5.

We write $X[A/x]$ for the substitution of A for x in X. It is defined in the usual capture-avoiding way (see, *e.g.*, [Jac99] for details).

The substitution rule for types

$$\frac{\Gamma \vdash M\colon \sigma \qquad \Gamma, x\colon \sigma, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[M/x] \vdash \mathcal{J}[M/x]}\ \textbf{substitution for types}$$

is interpreted as follows. To simplify the notation we will assume $\Delta = y\colon \tau$ — the more general case is not more difficult, just even more notationally cumbersome. There two cases, depending on the form of $\mathcal{J}$.

If $\mathcal{J} = \sigma'\colon \mathsf{Type}$ and

- Γ is interpreted by $I = (X_I, A_I, E_I)$
- $\Gamma \vdash \sigma\colon \mathsf{Type}$ is interpreted by $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$
- $\Gamma \vdash M\colon \sigma$ is interpreted by a section $M\colon I \to \{\mathcal{X}\}$ in $\mathbf{Mod(ALat)}$
- $\Gamma, x\colon \sigma \vdash \tau\colon \mathsf{Type}$ is interpreted by

$$\mathcal{Y} = (\{\mathcal{X}\}, B, (Y_{(i,x)}, E_{(i,x)})_{(i,x) \in \coprod_{i \in X_I} X_i})$$

- $\Gamma, x\colon \sigma, \Delta \vdash \sigma'\colon \mathsf{Type}$ is interpreted by

$$\mathcal{Z} = (\{\mathcal{Y}\}, C, (Z_{((i,x),y)}, E_{((i,x),y)})_{((i,x),y) \in \coprod_{(i,x)\in \cdots} Y_i})$$

then $\Gamma, \Delta[M/x] \vdash \mathcal{J}[M/x]$ is interpreted by

$$\mathcal{Z}^* = (W, C, (Z_{(m(i),y)}, E_{(m(i),y)})_{(i,y) \in \coprod_{i \in X_I} Y_{m(i)}}),$$

where $W = (\coprod_{i \in X_I} Y_{m(i)}, A_I \times B, E)$ with $E(i, y) = E_I(i) \wedge E_{m(i)}(y)$.

If $\mathcal{J} = M'\colon \sigma'$ and we assume the same interpretation for the fixed components as above and

- $\Gamma, x\colon \sigma, \Delta \vdash M'\colon \sigma'$ is interpreted by a morphism from the terminal object over $\{\mathcal{Y}\}$ to $\mathcal{Z}$ or equivalently a section $M'\colon \{\mathcal{Y}\} \to \{\mathcal{Z}\}$ of the form $((i, x), y) \mapsto (((i, x), y), z_{((i,x),y)})$

then $\Gamma, \Delta[M/x] \vdash \mathcal{J}[M/x]$ is interpreted by the section $W \to \{\mathcal{Z}^*\}$ mapping (i, y) to $z_{(m(i),y)}$.

The contraction rule for types

$$\frac{\Gamma, x\colon \sigma, y\colon \sigma, \Delta \vdash \mathcal{J}}{\Gamma, x\colon \sigma, \Delta[x/y] \vdash \mathcal{J}[x/y]}\ \textbf{contraction for types}$$

is interpreted as follows. To simplify the notation we will assume that $\Delta = z\colon \tau$ with $z \neq y$. Again there are two cases, depending on the form of $\mathcal{J}$.

If $\mathcal{J} = \tau'\colon \mathsf{Type}$ and

- Γ is interpreted by $I = (X_I, A_I, E_I)$
- $\Gamma \vdash \sigma\colon \mathsf{Type}$ is interpreted by $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$ so that

$$(\Gamma, x\colon \sigma, y\colon \sigma)\colon \mathsf{Ctx}$$

  is interpreted by

$$\mathcal{Y} = \left(\{\mathcal{X}\}, A, (X_i, E_i)_{(i,x) \in \coprod_{i \in X_I} X_i}\right)$$

- $\Gamma, x\colon \sigma, y\colon \sigma \vdash \tau\colon \mathsf{Type}$ is interpreted by

$$\mathcal{Z} = \left(\{\mathcal{Y}\}, B, (Z_{((i,x),x')}, E_{((i,x),x')})_{((i,x),x') \in \coprod_{(i,x) \in \coprod_{i \in X_I} X_i} X_i}\right)$$

- $\Gamma, x\colon \sigma, y\colon \sigma, z\colon \tau \vdash \tau'\colon \mathsf{Type}$ is interpreted by

$$\mathcal{W} = \left(\{\mathcal{Z}\}, C, (W_{(((i,x),x'),z)}, E_{(((i,x),x'),z)})_{(((i,x),x'),z) \in \coprod_{((i,x),x') \in \cdots} Z_{((i,x),x')}}\right)$$

then $\Gamma, x\colon \sigma, \Delta[x/y] \vdash \mathcal{J}[x/y]$ is interpreted by

$$\left(V, C, (W_{(((i,x),x),z)}, E_{(((i,x),x'),z)})_{((i,x),z) \in |V|}\right),$$

where $V = \left(\coprod_{(i,x) \in \coprod_{i \in X_I} X_i} Z_{((i,x),x)}, A_I \times A \times B, E\right)$ with $E((i,x),z) = E_I(i) \wedge E_i(x) \wedge E_{((i,x),x)}(z)$.

If $\mathcal{J} = M\colon \tau'$ and we assume the same interpretation for the fixed components as above and

- $\Gamma, x\colon \sigma, y\colon \sigma, z\colon \tau \vdash M\colon \tau'$ is interpreted by a morphism from the terminal object over $\{\mathcal{Z}\}$ to $\mathcal{W}$ or equivalently as a section $M\colon \{\mathcal{Z}\} \to \{\mathcal{W}\}$ of the form $(((i,x),x'),z) \mapsto ((((i,x),x'),z), w_{(((i,x),x'),z)})$

then $\Gamma, x\colon \sigma, z\colon \tau \vdash \Delta[x/y] \vdash \mathcal{J}[x/y]$ is interpreted by the section mapping $((i,x),z)$ to $(((i,x),z), w_{(((i,x),x),z)})$.

The weakening rule for types

$$\frac{\Gamma \vdash \sigma\colon \mathsf{Type} \qquad \Gamma \vdash \mathcal{J}}{\Gamma, x\colon \sigma \vdash \mathcal{J}}\ \textbf{weakening for types}$$

is interpreted as follows.

If $\mathcal{J} = \tau\colon \mathsf{Type}$ and

- Γ is interpreted by $I = (X_I, A_I, E_I)$
- $\Gamma \vdash \sigma\colon \mathsf{Type}$ is interpreted by $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$
- $\Gamma \vdash \tau\colon \mathsf{Type}$ is interpreted by $\mathcal{Y} = (I, B, (Y_i, E_i)_{i \in X_I})$

then $\Gamma, x\colon \sigma \vdash \mathcal{J}$ is interpreted as $\mathcal{Z} = (\{\mathcal{X}\}, B, (Y_i, E_i)_{(i,x) \in \coprod_{i \in X_I} X_i})$.

If $\mathcal{J} = M\colon \tau$ and we assume the same interpretation for the fixed components as above and

- $\Gamma \vdash M\colon \tau$ is interpreted by a section $I \to \{\mathcal{Y}\}$ mapping i to (i, y_i)

then $\Gamma, x\colon \sigma \vdash \mathcal{J}$ is interpreted as the section $(i,x) \mapsto ((i,x), y_i)\colon \{\mathcal{X}\} \to \{\mathcal{Z}\}$, where $\mathcal{Z}$ is as above.

The exchange rule for types

$$\frac{\Gamma, x\colon \sigma, y\colon \tau, \Delta \vdash \mathcal{J} \qquad x \text{ not free in } \tau}{\Gamma, y\colon \tau, x\colon \sigma, \Delta \vdash \mathcal{J}}\ \textbf{exchange for types}$$

is interpreted as follows. For notational simplicity we assume $\Delta = \emptyset$. If $\mathcal{J} = \tau'\colon \mathsf{Type}$ and

- Γ is interpreted by $I = (X_I, A_I, E_I)$
- $\Gamma \vdash \sigma\colon \mathsf{Type}$ is interpreted by $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$
- $\Gamma \vdash \tau\colon \mathsf{Type}$ is interpreted by $\mathcal{Y} = (I, B, (Y_i, E_i)_{i \in X_I})$

- $\Gamma, x\colon \sigma, y\colon \tau \vdash \tau'\colon \mathsf{Type}$ is interpreted by

$$\mathcal{Z} = \left(\coprod_{(i,x)\in\coprod_{i\in X_I} X_i} Y_i, C, (Z_{((i,x),y)}, E_{((i,x),y)})_{((i,x),y)\in\cdots}\right)$$

then $\Gamma, y\colon \tau, x\colon \sigma \vdash \tau'\colon \mathsf{Type}$ is interpreted by

$$\left(\coprod_{(i,y)\in\coprod_{i\in X_I} Y_i} X_i, C, (Z_{((i,y),x)}, E_{((i,y),x)})_{((i,y),x)\in\cdots}\right).$$

If $\mathcal{J} = M\colon \tau'$ the interpretation is similarly obvious.

Having covered the basic structural rules we now go on to consider the different type formers and their introduction and elimination rules.

Rules for Unit Type The singleton (unit) type

$$\frac{}{\vdash 1\colon \mathsf{Type}}$$

is interpreted as the terminal object in the fibre $\mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_1$ over the terminal object in $\mathbf{Mod}(\mathbf{ALat})$; explicitly, as the object

$$\left(1_{\mathbf{Mod}(\mathbf{ALat})}, 1_{\mathbf{ALat}}, (1_{\mathbf{Set}}, \top_1)_{*\in|1|}\right)$$

with $1_{\mathbf{Mod}(\mathbf{ALat})} = (1_{\mathbf{Set}}, 1_{\mathbf{ALat}}, \top_1)$ and $\top_1 = \lambda x.\ \{id_{1_{\mathbf{ALat}}}\}$.

The term

$$\frac{}{\vdash \langle\rangle\colon 1}$$

is interpreted as the identity arrow on $1 \in \mathbf{Mod}(\mathbf{ALat})$ (or, equivalently, as the identity arrow on $1 \in \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_1$.

The associated term conversion rule for the unit type is

$$\frac{\Gamma \vdash M\colon 1}{\Gamma \vdash M = \langle\rangle\colon 1}$$

Rules for Dependent Product Types The formation rule for dependent product

$$\frac{\Gamma, x\colon \sigma \vdash \tau\colon \mathsf{Type}}{\Gamma \vdash \Pi x\colon \sigma.\ \tau\colon \mathsf{Type}}$$

is interpreted as follows. If

- Γ is interpreted by $I = (X_I, A_I, E_I)$
- $\Gamma \vdash \sigma\colon \mathsf{Type}$ is interpreted by $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$
- $\Gamma, x\colon \sigma \vdash \tau\colon \mathsf{Type}$ is interpreted by

$$\mathcal{Y} = (\{\mathcal{X}\}, B, (Y_{(i,x)}, E_{(i,x)})_{(i,x) \in \coprod_{i \in X_I} X_i})$$

then $\Gamma \vdash \Pi x\colon \sigma.\ \tau\colon \mathsf{Type}$ is interpreted by $\mathcal{Z} = (I, B^A, (U_i, E_i'')_{i \in X_I})$, where

$$U_i = \{\, f\colon X_i \to \bigcup_{x \in X_i} Y_{(i,x)} \mid \forall x \in X_i.\ f(x) \in Y_{(i,x)} \text{ and } E_i''(f) \neq \emptyset \,\}$$

and

$$E_i''(f) = \{\, g \in B^A \mid \forall x \in X_i.\ \forall a \in E_i(x).\ g(a) \in E_{(i,x)}(f(x)) \,\}.$$

The introduction rule

$$\frac{\Gamma, x\colon \sigma \vdash M\colon \tau}{\Gamma \vdash \lambda x\colon \sigma.\ M\colon \Pi x\colon \sigma.\ \tau}$$

is interpreted as follows. If Γ; $(\Gamma \vdash \sigma\colon \mathsf{Type})$; $(\Gamma, x\colon \sigma \vdash \tau\colon \mathsf{Type})$; and $(\Pi x\colon \sigma.\ \tau\colon \mathsf{Type})$ are interpreted as above and

- $\Gamma, x\colon \sigma \vdash M\colon \tau$ is interpreted as a morphism from the terminal object over $\{\mathcal{X}\}$ to $\mathcal{Y}$ in the fibre $\mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_{\{\mathcal{X}\}}$ or, equivalently, as a section $M\colon \{\mathcal{X}\} \to \{\mathcal{Y}\}$ mapping (i,x) to $((i,x), y_{(i,x)})$

then $\Gamma \vdash \lambda x\colon \sigma.\ M\colon \Pi x\colon \sigma.\ \tau$ is interpreted as the section $I \to \{\mathcal{Z}\}$ (with $\mathcal{Z}$ as above) mapping i to $(i, x \in X_i \mapsto y_{(i,x)})$.

The elimination rule

$$\frac{\Gamma \vdash M\colon \Pi x\colon \sigma.\ \tau \qquad \Gamma \vdash N\colon \sigma}{\Gamma \vdash MN\colon \tau[N/x]}$$

is interpreted as follows. If Γ; $(\Gamma \vdash \sigma\colon \mathsf{Type})$; $(\Gamma, x\colon \sigma \vdash \tau\colon \mathsf{Type})$; and $(\Pi x\colon \sigma.\ \tau\colon \mathsf{Type})$ are interpreted as above, and

- $\Gamma \vdash M \colon \Pi x \colon \sigma.\ \tau$ is interpreted as the section $M \colon I \to \{\mathcal{Z}\}$ mapping $i \in X_I$ to (i, f_i)
- $\Gamma \vdash N \colon \sigma$ is interpreted as the section $I \to \{\mathcal{X}\}$ mapping $i \in X_I$ to (i, x_i)

then $\Gamma \vdash MN \colon \tau[N/x]$ is interpreted as the section from I to $\{\mathcal{W}\}$ given by $i \mapsto (i, f_i(x_i))$, where $\mathcal{W} = (I, B, (Y_{(i,x_i)}, E_{(i,x_i)})_{i \in X_I})$

There are the usual (β)- and (η)-conversions for dependent product types (we write them in a simplified form, omitting contexts *etc.*):

$$(\lambda x \colon \sigma.\ M)N = M[N/x]$$
$$\lambda x \colon \sigma.\ Mx = M$$

with the usual proviso that in the (η)-conversion the variable x is not allowed to occur free in M.

Rules for Strong Dependent Sum Types The formation rule for dependent sum

$$\frac{\Gamma, x \colon \sigma \vdash \tau \colon \mathsf{Type}}{\Gamma \vdash \Sigma x \colon \sigma.\ \tau \colon \mathsf{Type}}$$

is interpreted as follows. If

- Γ is interpreted by $I = (X_I, A_I, E_I)$
- $\Gamma \vdash \sigma \colon \mathsf{Type}$ is interpreted by $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$
- $\Gamma, x \colon \sigma \vdash \tau \colon \mathsf{Type}$ is interpreted by
$$\mathcal{Y} = (\{\mathcal{X}\}, B, (Y_{(i,x)}, E_{(i,x)})_{(i,x) \in \coprod_{i \in X_I} X_i})$$

then $\Gamma \vdash \Sigma x \colon \sigma.\ \tau \colon \mathsf{Type}$ is interpreted as

$$\mathcal{Z} = \big(I, A \times B, (Z_i, E_i'')_{i \in X_I}\big),$$

where $Z_i = \{\, (x, y) \mid x \in X_I \text{ and } y \in Y_{(i,x)} \,\}$, and $E_i''(x, y) = E_i(x) \wedge E_{(i,x)}(y) = \{\, \langle a, b \rangle \mid a \in E_i(x) \text{ and } b \in E_{(i,x)}(y) \,\}$.

The introduction rule

$$\frac{\Gamma \vdash \sigma \colon \mathsf{Type} \qquad \Gamma, x \colon \sigma \vdash \tau \colon \mathsf{Type}}{\Gamma, x \colon \sigma, y \colon \tau \vdash \langle x, y \rangle \colon \Sigma x \colon \sigma.\ \tau}$$

is interpreted as follows. Suppose that Γ; $(\Gamma \vdash \sigma\colon \mathsf{Type})$; $(\Gamma, x\colon \sigma \vdash \tau\colon \mathsf{Type})$; and $(\Gamma \vdash \Sigma x\colon \sigma.\ \tau\colon \mathsf{Type})$ are interpreted as above. Let W be the interpretation of $(\Gamma, x\colon \sigma, y\colon \tau)\colon \mathsf{Ctx}$, that is,

$$W = \big(\textstyle\coprod_{(i,x)\in\coprod_{i\in X_I} X_i} Y_{(i,x)}, (A_I \times A) \times B, E\big)$$

with $E((i,x),y) = E_I(i) \wedge E_i(x) \wedge E_{(i,x)}(y)$.

Let $\mathcal{Z}^*$ be $\mathcal{Z}$ "reindexed to the context $\Gamma, x\colon \sigma, y\colon \tau$," that is,

$$\mathcal{Z}^* = \big(W, A \times B, (Z_i, E_i'')_{((i,x),y)\in|W|}\big).$$

Then $\Gamma, x\colon \sigma, y\colon \tau \vdash \langle x, y\rangle\colon \Sigma x\colon \sigma.\ \tau$ is interpreted as the section

$$((i,x),y) \mapsto \big(((i,x),y),(x,y)\big)$$

from W to $\{\mathcal{Z}^*\}$, or, equivalently, as the morphism

$$(id, (* \mapsto (x,y))_{((i,x),y)\in|W|})$$

from the terminal object to $\mathcal{Z}^*$ in the fibre over W.

The (strong) elimination rule

$$\frac{\Gamma, z\colon \Sigma x\colon \sigma.\ \tau \vdash \rho\colon \mathsf{Type} \qquad \Gamma, x\colon \sigma, y\colon \tau \vdash Q\colon \rho[\langle x,y\rangle/z]}{\Gamma, z\colon \Sigma x\colon \sigma.\ \tau \vdash (\mathsf{unpack}\ z\ \mathsf{as}\ \langle x,y\rangle\ \mathsf{in}\ Q)\colon \rho}\ (\mathit{strong})$$

is interpreted as follows. (The variables x and y are bound in the sum elimination term $\mathsf{unpack}\ z\ \mathsf{as}\ \langle x,y\rangle\ \mathsf{in}\ Q$.) Suppose that Γ; $(\Gamma \vdash \sigma\colon \mathsf{Type})$; $(\Gamma, x\colon \sigma \vdash \tau\colon \mathsf{Type})$; and $(\Gamma \vdash \Sigma x\colon \sigma.\ \tau\colon \mathsf{Type})$ are interpreted as above. Then $(\Gamma, z\colon \Sigma x\colon \sigma.\ \tau)\colon \mathsf{Ctx}$ is interpreted as $V = (\coprod_{i\in X_I} Z_i, A_I \times (A \times B), E)$ with $E(i,(x,y)) = E_I(i) \wedge E_i(x) \wedge E_{(i,x)}(y)$. If

- $\Gamma, z\colon \Sigma x\colon \sigma.\ \tau \vdash \rho\colon \mathsf{Type}$ is interpreted as

$$\mathcal{V} = \big(V, C, (V_{(i,(x,y))}, E_{(i,(x,y))})_{(i,(x,y))\in|V|}\big)$$

- $\Gamma, x\colon \sigma, y\colon \tau \vdash Q\colon \rho[\langle x,y\rangle/z]$ is interpreted as the section

$$((i,x),y) \mapsto \big(((i,x),y), v_{(i,(x,y))}\big)$$

from W to

$$\{\big(W, C, (V_{(i,(x,y))}, E_{(i,(x,y))})_{((i,x),y)\in|W|}\big)\},$$

where W as above is the interpretation of $(\Gamma, x\colon \sigma, y\colon \tau)\colon \mathsf{Ctx}$

then $\Gamma, z\colon \Sigma x\colon \sigma.\ \tau \vdash (\mathsf{unpack}\ z\ \mathsf{as}\ \langle x, y\rangle\ \mathsf{in}\ Q)\colon \rho$ is interpreted as the section

$$(i,(x,y)) \mapsto \big((i,(x,y)), v_{(i,(x,y))}\big)$$

from V to $\{\mathcal{V}\}$.

There are the usual (β)- and (η)-conversions:

$$\begin{aligned} \mathsf{unpack}\ \langle M, N\rangle\ \mathsf{as}\ \langle x, y\rangle\ \mathsf{in}\ Q &= Q[M/x, N/y] \\ \mathsf{unpack}\ P\ \mathsf{as}\ \langle x, y\rangle\ \mathsf{in}\ Q[\langle x, y\rangle/z] &= Q[P/z]. \end{aligned}$$

The strong elimination rule for sums can equivalently be formulated via the perhaps more familiar rules

$$\frac{\Gamma \vdash P\colon \Sigma x\colon \sigma.\ \tau}{\Gamma \vdash \pi P\colon \sigma} \qquad \frac{\Gamma \vdash P\colon \Sigma x\colon \sigma.\ \tau}{\Gamma \vdash \pi' P\colon \tau[\pi P/x]}$$

with conversions $\pi\langle M, N\rangle = M$, $\pi'\langle M, N\rangle = N$, and $\langle \pi P, \pi' P\rangle = P$. See [Jac99, Section 10.1] for the correspondence between the two formulations.

Structural Rules for Dependent Predicate Logic There are the following structural rules for dependent predicate logic.

$$\frac{\Gamma \vdash \psi\colon \mathsf{Prop}}{\Gamma \mid \psi \vdash \psi}\ \textbf{identity}$$

$$\frac{\Gamma \mid \Theta \vdash \varphi \qquad \Gamma \mid \Theta', \varphi \vdash \psi}{\Gamma \mid \Theta, \Theta' \vdash \psi}\ \textbf{cut}$$

$$\frac{\Gamma \mid \Theta \vdash \psi \qquad \Gamma \vdash \varphi\colon \mathsf{Prop}}{\Gamma \mid \Theta, \varphi \vdash \psi}\ \textbf{weakening for propositions}$$

$$\frac{\Gamma \mid \Theta, \varphi, \varphi \vdash \psi}{\Gamma \mid \Theta, \varphi \vdash \psi}\ \textbf{contraction for propositions}$$

$$\frac{\Gamma \mid \Theta, \varphi, \chi, \Theta' \vdash \psi}{\Gamma \mid \Theta, \chi, \varphi, \Theta' \vdash \psi}\ \textbf{exchange for propositions}$$

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma, x : \sigma, \Delta \mid \Theta \vdash \psi}{\Gamma, \Delta[M/x] \mid \Theta[M/x] \vdash \psi[M/x]} \textbf{ substitution for propositions}$$

Given what we have said before, to understand the interpretation of these structural rules we only need to explicate how substitution of terms for variables in propositions is interpreted. We go through the interpretation of the substitution rule for propositions in detail. Suppose for notational simplicity that $\Delta = (y : \tau)$. Then if

- Γ is interpreted as $I = (X_I, A_I, E_I)$.
- $\Gamma \vdash \sigma : \mathsf{Type}$ is interpreted as $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$.
- $\Gamma \vdash M : \sigma$ is interpreted as a section $I \to \{\mathcal{X}\}$ mapping i to (i, x_i).
- $(\Gamma, x : \sigma) : \mathsf{Ctx}$ is interpreted as $\{\mathcal{X}\}$.
- $\Gamma, x : \sigma \vdash \tau : \mathsf{Type}$ is interpreted as

$$\mathcal{Y} = \big(\{\mathcal{X}\}, B, (Y_{(i,x)}, E_{(i,x)})_{(i,x) \in \coprod_{i \in X_I} X_i}\big).$$

- $(\Gamma, x : \sigma, \Delta) : \mathsf{Ctx}$ is interpreted as $\{\mathcal{Y}\}$.
- $\Gamma, x : \sigma, \Delta \vdash \Theta : \mathsf{Prop}$ (we here think of Θ as a proposition, the conjunction of its constituent propositions) is interpreted as the predicate

$$(A_\Theta, \Theta : |\{\mathcal{Y}\}| \to PA_\Theta).$$

- $\Gamma, x : \sigma, \Delta \vdash \psi : \mathsf{Prop}$ is interpreted as predicate $(A_\psi, \psi : |\{\mathcal{Y}\}| \to PA_\psi)$.
- $\Gamma \vdash \tau[M/x] : \mathsf{Type}$ is interpreted as "the reindexing of $\mathcal{Y}$ along M", that is,

$$\mathcal{Y}^* = \big(I, B, (Y_{(i,x_i)}, E_{(i,x_i)})_{i \in X_I}\big).$$

- $(\Gamma, \tau[M/x]) : \mathsf{Ctx}$ is interpreted as $\{\mathcal{Y}^*\}$.
- $(\Gamma, \tau[M/x]) \vdash \Theta[M/x] : \mathsf{Prop}$ is interpreted as the reindexing of ψ along the map $(i, y) \mapsto ((i, x_i), y) : \{\mathcal{Y}^*\} \to \{\mathcal{Y}\}$, that is, as $\Theta' = (i, y) \mapsto \Theta((i, x_i), y)$.

- $(\Gamma, \tau[M/x]) \vdash \psi[M/x] \colon \mathsf{Prop}$ is interpreted as the reindexing of ψ along the map $(i,y) \mapsto ((i,x_i),y) \colon \{\mathcal{Y}^*\} \to \{\mathcal{Y}\}$, that is, as $\psi' = (i,y) \mapsto \psi((i,x_i),y)$.

and $\Gamma, x \colon \sigma, \Delta \mid \Theta \vdash \psi$ is interpreted as the inequality $\Theta \leq \psi$ in the fibre $\mathbf{UFam(ALat)}_{\{\mathcal{Y}\}}$ then $\Gamma, \Delta[M/x] \mid \Theta[M/x] \vdash \psi[M/x]$ is interpreted as the inequality $\Theta' \leq \psi'$ in $\mathbf{UFam(ALat)}_{\{\mathcal{Y}^*\}}$.

Formation Rules for Propositions In the following explanation of the interpretation of the formation rules for propositions we always assume that

- $\Gamma \colon \mathsf{Ctx}$ is derivable and is interpreted by $I = (X_I, A_I, E_I)$
- $\Gamma \vdash \varphi \colon \mathsf{Prop}$ is derivable and is interpreted by $\varphi \colon X_I \to PA_\varphi$
- $\Gamma \vdash \psi \colon \mathsf{Prop}$ is derivable and is interpreted by the subobject $\psi \colon X_I \to PA_\psi$

For brevity, in many cases we do not show the formation of the propositions as formal rules. The propositions and their interpretations are described as follows.

- The atomic proposition

$$\frac{\Gamma \vdash M \colon \sigma \qquad \Gamma \vdash N \colon \sigma}{\Gamma \vdash (M =_\sigma N) \colon \mathsf{Prop}}$$

  is interpreted as follows. Suppose that $\Gamma \vdash \sigma \colon \mathsf{Type}$ is interpreted as $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$ and that M is interpreted as a section $i \mapsto (i, m_i) \colon I \to \{\mathcal{X}\}$ and that N is interpreted as a section $i \mapsto (i, n_i) \colon I \to \{\mathcal{X}\}$. Then $\Gamma \vdash (M =_\sigma N) \colon \mathsf{Prop}$ is interpreted as the predicate $X_I \to PA_I$ given by

$$i \mapsto \begin{cases} E_I(i) & \text{if } m_i = n_i, \\ \emptyset & \text{otherwise.} \end{cases}$$

- $\Gamma \vdash \top \colon \mathsf{Prop}$ is interpreted by the predicate $X_I \mapsto P1_{\mathbf{ALat}}$ given by $i \mapsto \{id_1\}$.
- $\Gamma \vdash \varphi \wedge \psi \colon \mathsf{Prop}$ is interpreted by the predicate $X_I \to P(A_\varphi \times A_\psi)$ given by $i \mapsto \varphi(i) \wedge \psi(i) = \{\, \langle a, a' \rangle \mid a \in \varphi(i) \text{ and } a' \in \psi(i) \,\}$.
- $\Gamma \vdash \bot \colon \mathsf{Prop}$ is interpreted by the predicate $X_I \to P1_{\mathbf{ALat}}$ given by $i \mapsto \emptyset$.

- $\Gamma \vdash \varphi \vee \psi \colon \mathsf{Prop}$ is interpreted by the predicate $X_I \to (\Sigma \times \Sigma) \times A \times B$, where Σ is the algebraic lattice $\bot \leq \top$, given by

$$i \mapsto \big(\{(\bot, \top)\} \times \varphi(i) \times \{\bot_{A_\psi}\}\big) \cup \big(\{(\top, \bot)\} \times \{\bot_{A_\varphi}\} \times \psi(i)\big)$$

- $\Gamma \vdash \varphi \supset \psi \colon \mathsf{Prop}$ is interpreted by the predicate $X_I \to P((A_\psi)^{(A_\varphi)})$ given by

$$i \mapsto \{\, g \in (A_\psi)^{(A_\varphi)} \mid \forall a \in \varphi(i).\ g(a) \in \psi(i) \,\}.$$

- $\Gamma \vdash \neg\varphi \colon \mathsf{Prop}$ is an abbreviation for $\Gamma \vdash \varphi \supset \bot \colon \mathsf{Prop}$. Working out the interpretation according to the above we find that $\Gamma \vdash \neg\varphi \colon \mathsf{Prop}$ is interpreted by the predicate $X_I \to P1_{\mathbf{ALat}}$ given by

$$i \mapsto \begin{cases} \emptyset & \text{if } \varphi(i) \neq \emptyset, \\ \{id_1\} & \text{if } \varphi(i) = \emptyset. \end{cases}$$

  Hence $\Gamma \vdash \neg\neg\varphi \colon \mathsf{Prop}$ is interpreted as

$$i \mapsto \begin{cases} \emptyset & \text{if } \varphi(i) = \emptyset, \\ \{id_1\} & \text{if } \varphi(i) \neq \emptyset. \end{cases}$$

  Under the equivalence given by Proposition 4.4.3, this is a regular subobject of I, so in $\mathbf{Mod(ALat)}$, *the regular subobjects of I correspond to double-negation closed subobjects of I.*

The formation rule for the universal quantifier

$$\frac{\Gamma, x \colon \sigma \vdash \varphi \colon \mathsf{Prop}}{\Gamma \vdash \forall x \colon \sigma.\ \varphi \colon \mathsf{Prop}}$$

is interpreted as follows. If

- $\Gamma \vdash \sigma \colon \mathsf{Type}$ is interpreted as $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$
- $\Gamma, x \colon \sigma \vdash \varphi \colon \mathsf{Prop}$ is interpreted as a predicate $\coprod_{i \in X_I} X_i \to PA_\varphi$

then $\Gamma \vdash \forall x \colon \sigma.\ \varphi \colon \mathsf{Prop}$ is interpreted as the predicate $X_I \to P((A_\varphi)^{(A_I)})$ given by

$$i \mapsto \bigcap_{x \in X_i} \{\, g \in ((A_\varphi)^{(A_I)}) \mid \forall a \in E_i(x).\ g(a) \in \varphi(i, x) \,\}.$$

The formation rule for the existential quantifier

$$\frac{\Gamma, x : \sigma \vdash \varphi : \mathsf{Prop}}{\Gamma \vdash \exists x : \sigma.\ \varphi : \mathsf{Prop}}$$

is interpreted as follows. If Γ; $(\Gamma \vdash \sigma : \mathsf{Type})$; and $(\Gamma, x : \sigma \vdash \varphi : \mathsf{Prop})$ are interpreted as above in the description of the formation rule for universal quantification, then $\Gamma \vdash \exists x : \sigma.\ \varphi : \mathsf{Prop}$ is interpreted as the predicate $X_I \to P(A_I \times A_\varphi)$ given by

$$i \mapsto \bigcup_{x \in X_i} (E_i(x) \wedge \varphi(i, x)).$$

Logical Rules for Dependent Predicate Logic The logical rules are shown in Figure A.1, copied from [Jac99, Section 4.1]. Note that propositional equality $(M =_\sigma M')$ includes term conversion, so internal equality in the logic includes external equality. See [Jac99, Section 4.1] for further comments and for equivalent adjoint formulations of some of the rules.

Dependent Subset Types The formation rule for dependent subset types

$$\frac{\Gamma, x : \sigma \vdash \varphi : \mathsf{Prop}}{\Gamma \vdash \{\, x : \sigma \mid \varphi \,\} : \mathsf{Type}}$$

is interpreted as follows. If

- $\Gamma : \mathsf{Ctx}$ is interpreted by $I = (X_I, A_I, E_I)$
- $\Gamma \vdash \sigma : \mathsf{Type}$ is interpreted by $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$
- $\Gamma, x : \sigma \vdash \varphi : \mathsf{Prop}$ is interpreted by predicate $\varphi : \coprod_{i \in X_I} X_i \to P A_\varphi$

then $\Gamma \vdash \{\, x : \sigma \mid \varphi \,\} : \mathsf{Type}$ is interpreted by

$$(I, A_\varphi, (Z_i, E_i')_{i \in X_I}) \in \mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_I,$$

where $Z_i \stackrel{\text{def}}{=} \{\, x \in X_i \mid \varphi(i, x) \neq \emptyset \,\}$, and $E_i(x) \stackrel{\text{def}}{=} \varphi(i, x)$.

The introduction rule

$$\frac{\Gamma, x : \sigma \vdash \varphi : \mathsf{Prop} \qquad \Gamma, y : \tau \vdash M : \sigma \qquad \Gamma, y : \tau \mid \emptyset \vdash \varphi[M/x]}{\Gamma, y : \tau \vdash \mathsf{i}(M) : \{\, x : \sigma \mid \varphi \,\}}$$

is interpreted as follows. Suppose Γ; $(\Gamma \vdash \sigma : \mathsf{Type})$; and $(\Gamma, x : \sigma \vdash \varphi : \mathsf{Prop})$ are interpreted as above. If

$$\frac{}{\Gamma \mid \Theta \vdash \top}$$

$$\frac{\Gamma \mid \Theta \vdash \varphi \qquad \Gamma \mid \Theta \vdash \psi}{\Gamma \mid \Theta \vdash \varphi \wedge \psi}$$

$$\frac{\Gamma \mid \Theta \vdash \varphi \wedge \psi}{\Gamma \mid \Theta \vdash \varphi}$$

$$\frac{\Gamma \mid \Theta \vdash \psi}{\Gamma \mid \Theta \vdash \varphi \vee \psi}$$

$$\frac{\Gamma \mid \Theta, \varphi \vdash \psi}{\Gamma \mid \Theta \vdash \varphi \supset \psi}$$

$$\frac{\Gamma, x : \sigma \mid \Theta \vdash \psi}{\Gamma \mid \Theta \vdash \forall x : \sigma.\ \psi}$$
(x not free in Θ)

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \mid \Theta \vdash \psi[M/x]}{\Gamma \mid \Theta \vdash \exists x : \sigma.\ \psi}$$

$$\frac{\Gamma \vdash M = M' : \sigma}{\Gamma \mid \Theta \vdash M =_\sigma M'}$$

$$\frac{\Gamma \mid \Theta \vdash M =_\sigma M'}{\Gamma \mid \Theta \vdash M' =_\sigma M}$$

$$\frac{}{\Gamma \mid \Theta, \bot \vdash \psi}$$

$$\frac{\Gamma \mid \Theta \vdash \varphi \wedge \psi}{\Gamma \mid \Theta \vdash \varphi}$$

$$\frac{\Gamma \mid \Theta \vdash \varphi}{\Gamma \mid \Theta \vdash \varphi \vee \psi}$$

$$\frac{\Gamma \mid \Theta, \varphi \vdash \chi \qquad \Gamma \mid \Theta, \psi \vdash \chi}{\Gamma \mid \Theta, \varphi \vee \psi \vdash \chi}$$

$$\frac{\Gamma \mid \Theta \vdash \varphi \supset \psi \qquad \Gamma \mid \Theta \vdash \varphi}{\Gamma \mid \Theta \vdash \psi}$$

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \mid \Theta \vdash \forall x : \sigma.\ \psi}{\Gamma \mid \Theta \vdash \psi[M/x]}$$

$$\frac{\Gamma \mid \Theta \vdash \exists x : \sigma.\ \psi \qquad \Gamma, x : \sigma \mid \Theta', \psi \vdash \chi}{\Gamma \vdash \Theta, \Theta' \vdash \chi}$$
(x not free in Θ', χ)

$$\frac{\Gamma \mid \Theta \vdash M =_\sigma M' \qquad \Gamma \mid \Theta \vdash M' =_\sigma M''}{\Gamma \mid \Theta \vdash M =_\sigma M''}$$

$$\frac{\Gamma \mid \Theta \vdash M =_\sigma M' \qquad \Gamma \mid \Theta \vdash \psi[M/x]}{\Gamma \mid \Theta \vdash \psi[M'/x]}$$

Figure A.1: Logical Rules for Dependent Predicate Logic

- $\Gamma \vdash \tau\colon \mathsf{Type}$ is interpreted as $\mathcal{Y} = (I, B, (Y_i, E_i)_{i \in X_I})$.
- $\Gamma, y\colon \tau \vdash \sigma\colon \mathsf{Type}$ is interpreted as "$\mathcal{X}$ reindexed to $(\Gamma, y\colon \tau)\colon \mathsf{Ctx}$," that is,

$$\mathcal{X}^* = (\{\mathcal{Y}\}, A, (X_i, E_i)_{(i,x) \in \coprod_{i \in X_I} Y_i}).$$

- $\Gamma, y\colon \tau \vdash M\colon \sigma$ is interpreted by a section

$$(i, y) \mapsto ((i, y), x_{(i,y)})\colon \{\mathcal{Y}\} \to \{\mathcal{X}^*\}$$

  or, equivalently, as a morphism

$$(id, (y \mapsto x_{(i,y)})_{i \in X_I})\colon \mathcal{Y} \to \mathcal{X}^*$$

  in $\mathbf{UFam}(\mathbf{Mod}(\mathbf{ALat}))_I$.

- $\Gamma, y\colon \tau \vdash \varphi[M/x]\colon \mathsf{Prop}$ is interpreted as φ reindexed along $M\colon \{\mathcal{Y}\} \to \{\mathcal{X}\}$, that is, as

$$\varphi' = (i, y) \mapsto \varphi(i, x_{(i,y)})\colon \textstyle\coprod_{i \in X_I} Y_i \to PA_\varphi.$$

- $\Gamma, y\colon \tau \mid \emptyset \vdash \varphi[M/x]$ is interpreted as an inequality $\top_{\{\mathcal{Y}\}} \le \varphi'$ in $\mathbf{UFam}(\mathbf{ALat})_{\{\mathcal{Y}\}}$. Thus there is a realizer $g \in A_I \times B \to A_\varphi$ such that

$$\forall (i, y) \in \textstyle\coprod_{i \in X_I} Y_i.\ \forall a \in E_I(i).\ \forall b \in E_i(y).\ g(a, b) \in \varphi'(i, y).$$

then $\Gamma, y\colon \tau \vdash \mathsf{i}(M)\colon \{\, x\colon \sigma \mid \varphi \,\}$ is interpreted as the section $(i, y) \mapsto ((i, y), x_{(i,y)})$ (a well-defined morphism by the existence of realizer g).

The elimination rule

$$\frac{\Gamma \vdash N\colon \{\, x\colon \sigma \mid \varphi \,\}}{\Gamma \vdash \mathsf{o}(N)\colon \sigma}$$

is interpreted as follows. If Γ; $(\Gamma \vdash \sigma\colon \mathsf{Type})$; and $(\Gamma, x\colon \sigma \vdash \varphi\colon \mathsf{Prop})$ is interpreted as above and

- $\Gamma \vdash N\colon \{\, x\colon \sigma \mid \varphi \,\}$ is interpreted as a section

$$i \mapsto (i, z_i)\colon I \to \{(I, A_\varphi, (Z_i, E_i)_{i \in X_I})\}$$

then $\Gamma \vdash \mathsf{o}(N) : \sigma$ is interpreted as the section $i \mapsto (i, z_i) : I \to \{\mathcal{X}\}$.

The associated conversions are

$$\mathsf{o}(\mathsf{i}(M)) = M \qquad \text{and} \qquad \mathsf{i}(\mathsf{o}(N)) = N.$$

There are the following two associated logical rules.

$$\frac{\Gamma, x\colon \sigma \mid \Theta, \varphi \vdash \psi}{\Gamma, y\colon \{\, x\colon \sigma \mid \varphi \,\} \mid \Theta[\mathsf{o}(y)/x] \vdash \psi[\mathsf{o}(y)/x]}$$

and

$$\frac{\Gamma, y\colon \{\, x\colon \sigma \mid \varphi \,\} \mid \Theta[\mathsf{o}(y)/x] \vdash \psi[\mathsf{o}(y)/x]}{\Gamma, x\colon \sigma \mid \Theta, \varphi \vdash \psi}\ \textbf{full subset types}$$

Dependent Quotient Types The formation rule for dependent quotient types

$$\frac{\Gamma, x\colon \sigma, y\colon \sigma \vdash R(x,y) : \mathsf{Prop}}{\Gamma \vdash \sigma/R\colon \mathsf{Type}}$$

is interpreted as follows. If

- $\Gamma\colon \mathsf{Ctx}$ is interpreted as $I = (X_I, A_I, E_I)$
- $\Gamma \vdash \sigma\colon \mathsf{Type}$ is interpreted as $\mathcal{X} = (I, A, (X_i, E_i)_{i \in X_I})$
- $\Gamma, x\colon \sigma, y\colon \sigma \vdash R(x,y) : \mathsf{Prop}$ is interpreted as a predicate $R\colon |\{\{\mathcal{X}\}\}| \to PA_\varphi$ with

$$\{\{\mathcal{X}\}\} = \big(\textstyle\coprod_{(i,x) \in \coprod_{i \in X_I} X_i} X_i, (A_I \times A) \times A, E\big)$$

then $\Gamma \vdash \sigma/R\colon \mathsf{Type}$ is interpreted as

$$(I, A, (X_i/{\approx_i}, E_i')_{i \in X_I}),$$

where $\approx_i$ is the least equivalence relation on X_i containing $\sim_i$ with

$$x \sim_i x' \iff R((i,x), x') \neq \emptyset,$$

and where $E_i'([x]) = \bigcup_{x' \in [x]} E_i(x')$.

The introduction rule

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash [M]_R : \sigma/R}$$

is interpreted as follows If Γ and $\Gamma \vdash \sigma : \mathsf{Type}$ are interpreted as above and

- $\Gamma \vdash M : \sigma$ is interpreted as the section $i \mapsto (i, x_i) : I \to \{\mathcal{X}\}$

then $\Gamma \vdash [M]_R : \sigma/R$ is interpreted as the section $i \mapsto (i, [x_i])$.

The elimination rule

$$\frac{\Gamma \vdash \tau : \mathsf{Type} \qquad \Gamma, x : \sigma \vdash N : \tau \qquad \Gamma, x : \sigma, y : \sigma \mid R(x, y) \vdash Nx =_\tau Ny}{\Gamma, a : \sigma/R \vdash \mathsf{pick}\ x\ \mathsf{from}\ a\ \mathsf{in}\ Nx : \tau}$$

is interpreted as follows. Suppose that Γ and $\Gamma \vdash \sigma : \mathsf{Type}$ are interpreted as above and that

- $\Gamma \vdash \tau : \mathsf{Type}$ is interpreted as $\mathcal{Y} = (I, B, (Y_i, E_i)_{i \in X_I})$.
- $\Gamma, x : \sigma \vdash N : \tau$ is interpreted as the section

$$(i, x) \mapsto ((i, x), y_{(i,x)}) : I \to \{\mathcal{Y}\}.$$

- $\Gamma, x : \sigma, y : \sigma \mid R(x, y) \vdash Nx =_\tau Ny$ is interpreted in the usual way.

Then $\Gamma, a : \sigma/R \vdash \mathsf{pick}\ x\ \mathsf{from}\ a\ \mathsf{in}\ Nx : \tau$ is interpreted as the section

$$(i, [x]) \mapsto (i, y_{(i,x_i)}) : \{(I, A, (X_i/{\approx_i}, E_i')_{i \in X_I})\} \to \{\mathcal{Y}\},$$

where x is a chosen representative of $[x]$ and $((i, x), y_{(i,x_i)})$ is the result of applying the given section to (i, x).

The associated (β)- and (η)-conversions are

$$\begin{array}{ll} (\beta) & \mathsf{pick}\ x\ \mathsf{from}\ [M]_R\ \mathsf{in}\ N = N[M/x] \\ (\eta) & \mathsf{pick}\ x\ \mathsf{from}\ Q\ \mathsf{in}\ N[[x]_R/a] = N[Q/a]. \end{array}$$

In the (η)-conversion it is assumed that x does not occur free in N.

There is the following associated logical rule.

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash M' : \sigma}{\Gamma \mid R(M, M') \vdash [M]_R =_{\sigma/R} [M']_R}$$

(We do not have effective or full quotients, see Remark 4.3.2.)

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Symbols

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